Last time

Dielectrics (cont'd)

Molecules in dielectrics $\simeq$ small electric dipoles

Def. $\vec{P} = \dfrac{\text{electric dipole moment}}{\text{volume}}$ ~ polarization

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\rho_{free}(\vec{x}') - \vec{\nabla}'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

Def. $\boxed{\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}}$ electric displacement

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho_{free} \\ \vec{\nabla}\times E &= 0 \end{aligned}}$$

~ maxwell equations for electrostatics of dielectrics

Linear Isotropic Homogeneous (LIH) medium:

$$\vec{D} = \varepsilon \vec{E}$$

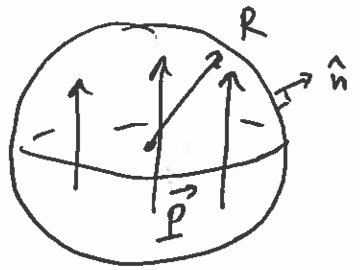
$\varepsilon \to \varepsilon_0 \Rightarrow$ vacuum

$\varepsilon \to \infty \Rightarrow$ conductor (perfect)

Boundary matching:

$\boxed{E_{1t} = E_{2t}}$ $\boxed{D_{1n} - D_{2n} = \sigma_f}$ $\boxed{P_{1n} - P_{2n} = -\sigma_{bound}}$

Example 1: uniformly polarized sphere:
find $\vec{E}$, $\vec{D}$



$$\Phi(\vec{x}) = -\frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\vec{\nabla}' \cdot \vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int da' \, \frac{\hat{n}' \cdot \vec{P}}{|\vec{x}-\vec{x}'|}$$

as $-\vec{\nabla}\cdot\vec{P}$ is like ρ (charge density), $\hat{n}\cdot\vec{P}$ is like σ (surf. dens.)

Spherical coordinates $\hat{n}' = (\sin\theta'\cos\varphi', \sin\theta'\sin\varphi', \cos\theta')$

$$\vec{P} = (\underset{x}{0}, \underset{y}{0}, \underset{z}{P})$$

$$\Rightarrow \hat{n}\cdot\vec{P} = P\cos\theta' = P \cdot P_1(\cos\theta') = P\sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta',\varphi')$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty}\sum_{m}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}\, Y_{\ell m}^*(\theta',\varphi')\, Y_{\ell m}(\theta\varphi)$$

now one of these is R, another one is r.

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\, P\sqrt{\frac{4\pi}{3}}\sum_{\ell,m}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}\int d\cos\theta'\, d\varphi' \cdot R^2$$

$$\cdot Y_{\ell m}^*(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)\, Y_{10}(\theta',\varphi') = \frac{1}{4\pi\varepsilon_0}\, P\sqrt{\frac{4\pi}{3}} \cdot \frac{4\pi}{3}\cdot R^2$$

$$\cdot \frac{r_<}{r_>^2}\, Y_{10}(\theta,\varphi) = \frac{P R^2}{\cancel{4\pi}\varepsilon_0}\frac{\cancel{4\pi}}{3}\cdot\frac{r_<}{r_>^2}\cos\theta = \frac{PR^2}{3\varepsilon_0}\frac{r_<}{r_>^2}\cos\theta$$

$$\Rightarrow \underset{r>R}{\Phi_{out} = \frac{P}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta}\, ; \quad \underset{r<R}{\Phi_{in} = \frac{P}{3\varepsilon_0}\, r\cos\theta}$$

$$\vec{E}_{out} = -\vec{\nabla}\,\Phi_{out} = \frac{R^3}{3\varepsilon_0}\left[\frac{3(n\cdot P)\,n - P}{r^3}\right]$$

$$\vec{E}_{in} = -\vec{\nabla}\,\Phi_{in} = -\frac{\vec{P}}{3\varepsilon_0}$$

$$\vec{D}_{out} = \varepsilon_0 \vec{E}_{out}\,, \qquad \vec{D}_{in} = \varepsilon_0 \vec{E}_{in} + \vec{P} = \frac{2}{3}\vec{P}.$$

as $\Phi_{out} = \dfrac{R^3}{3\varepsilon_0}\,\dfrac{\vec{P}\cdot\vec{r}}{r^3}$ ~ just a dipole potential

$\Phi_{in} = \dfrac{P}{3\varepsilon_0}\,z = \dfrac{\vec{P}\cdot\vec{r}}{3\varepsilon_0}$ ~ uniform $\vec{E}$ field potential

Net dipole moment of the sphere

$$\vec{p} = \int d^3x\,\vec{P} = \frac{4}{3}\pi R^3\,\vec{P}$$

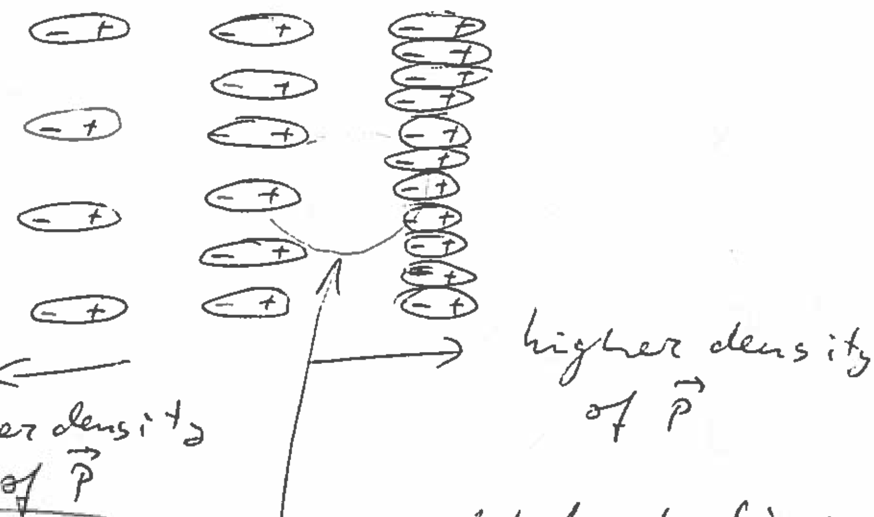
$\Rightarrow \Phi_{out} = \dfrac{1}{4\pi\varepsilon_0}\,\dfrac{\vec{p}\cdot\vec{r}}{r^3}$ ~ really a dipole potential

As the bound charge density is $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$\Rightarrow P_{1n} - P_{2n} = -\sigma_b$.

Why is $\rho_b = -\vec{\nabla}\cdot\vec{P}$?

Pictorially:



more likely to find more negative charges in a volume element

$\Downarrow$

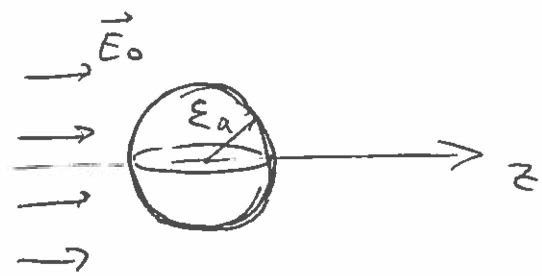
generates ρ_b!

Finally, if $\vec{D} = \varepsilon \vec{E}$

$\Rightarrow \vec{\nabla}\cdot\vec{D} = \varepsilon \vec{\nabla}\cdot\vec{E} = \rho_f \Rightarrow$

$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\varepsilon} \Rightarrow \boxed{\nabla^2 \Phi = -\frac{\rho_f}{\varepsilon}}$

Example 2: LIH dielectric sphere in external $\vec{E}$-field.



no free charges $\Rightarrow$

$\vec{\nabla}\cdot\vec{D} = 0$ inside & outside

$\vec{\nabla}\times\vec{E} = 0$ inside & outside

$\vec{D}_{out} = \varepsilon_0 \vec{E}_{out}$, $\vec{D}_{in} = \varepsilon \vec{E}_{in}$

$\Rightarrow$ as $\vec{\nabla}\times\vec{E} = 0 \Rightarrow$ $\vec{E}_{out} = -\vec{\nabla}\Phi_{out}$, $\vec{E}_{in} = -\vec{\nabla}\Phi_{in}$

$0 = \vec{\nabla}\cdot\vec{D}_{out} = \varepsilon_0 \vec{\nabla}\cdot\vec{E}_{out} = -\varepsilon_0 \nabla^2 \Phi_{out} \Rightarrow \nabla^2 \Phi_{out} = 0$

$$0 = \vec{\nabla}\cdot\vec{D}_{in} = \varepsilon\,\vec{\nabla}\cdot\vec{E}_{in} = -\varepsilon\nabla^2\Phi_{in} \Rightarrow \nabla^2\Phi_{in} = 0$$

$\Rightarrow$ We have $\nabla^2\Phi = 0$ everywhere (no free charges)

$\Rightarrow$ using the general solution of Laplace equation for problems with azimuthal symmetry in sphericall coordinates $\sum_\ell \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$

we write:

$$\Phi_{in} = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad , r < a$$

$$\Phi_{out} = \sum_{\ell=0}^{\infty} \left(B_\ell r^\ell + C_\ell r^{-\ell-1}\right) P_\ell(\cos\theta) \; , r > a.$$

We know that at $r \to \infty$ the potential should map onto that for the external field:

$$\Phi_{out}(r\to\infty) = -E_0 z = -E_0 r\cos\theta \Rightarrow$$

$\Rightarrow$ can fix B_ℓ's to write

$$\Phi_{out} = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} C_\ell r^{-\ell-1} P_\ell(\cos\theta).$$

Boundary conditions at the surface of the spere:

(1) $E_{in,t} = E_{out,t} \Rightarrow -\frac{1}{a}\frac{\partial \Phi_{in}}{\partial \theta}\Big|_{r=a} = -\frac{1}{a}\frac{\partial \Phi_{out}}{\partial \theta}\Big|_{r=a}$.

(2) $D_{in,n} = D_{out,n} \Rightarrow -\varepsilon \frac{\partial \Phi_{in}}{\partial r}\Big|_{r=a} = -\varepsilon_0 \frac{\partial \Phi_{out}}{\partial r}\Big|_{r=a}$

(1) $$\sum_{l=0}^{\infty} A_l a^l \frac{\partial}{\partial \theta} P_l(\cos\theta) = -E_0 a \frac{\partial}{\partial \theta} P_1(\cos\theta) + $$

$$+ \sum_{l=0}^{\infty} C_l a^{-l-1} \frac{\partial}{\partial \theta} P_l(\cos\theta)$$

associated Legendre function $P_l^m(x)$ with $m=1$.

as $P_l^1(\cos\theta) = \frac{\partial}{\partial \theta} P_l(\cos\theta)$ and P_l^1's are all orthogonal $\Rightarrow$

$$\begin{cases} A_l a^l = C_l a^{-l-1}, & l \neq 1 \\ A_1 a = -E_0 a + C_1 a^{-2} \end{cases}$$

(2) $$\varepsilon \sum_{l=0}^{\infty} A_l \cdot l \cdot a^{l-1} P_l(\cos\theta) = -\varepsilon_0 E_0 P_1(\cos\theta) +$$

$$+ \varepsilon_0 \sum_{l=0}^{\infty} C_l (-l-1) a^{-l-2} P_l(\cos\theta)$$

$\Rightarrow P_l$'s are orthogonal $\Rightarrow$