Last time | Electrostatic Energy in Dielectrics

$$\boxed{W = \int d^3x \int_0^D \delta \vec{D} \cdot \vec{E}}$$ ~ energy

$\Rightarrow$ in LIH medium get

$$\boxed{W = \frac{1}{2} \int d^3x \; \vec{E} \cdot \vec{D}}$$

Forces: $\boxed{F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)_Q}$ $\boxed{F_\xi = \left(\frac{\partial W}{\partial \xi}\right)_V}$.

Magnetostatics

Go back to the Maxwell equations (in SI units)

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} && \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \\ &\vec{\nabla}\cdot\vec{B} = 0 && \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \end{aligned}}$$

Also remember that the 4-vector of current, $J^\mu = (c\rho, \vec{J})$, is conserved: $\boxed{\partial_\mu J^\mu = 0}$

such that $\boxed{\dfrac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$.

We want to study magnetic field in a static case $\Rightarrow$ magnetostatics. Put $\vec{B}(\vec{x},t) = \vec{B}(\vec{x})$ and $\vec{E} = 0$ in Maxwell equations. We get

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{B} = 0 \\ &\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \end{aligned}}$$

Differential equations of magnetostatics.

Note that $\rho(\vec{x},t) \to \rho(\vec{x})$, $\vec{J}(\vec{x},t) \to \vec{J}(\vec{x})$.

The current conservation reduces to

$\vec{\nabla} \cdot \vec{J} = 0$

Returning to differential equations of magnetostatics: as $\vec{B} = \vec{\nabla} \times \vec{A}$ $\Rightarrow$ the 1^{st} eq'n $(\vec{\nabla} \cdot \vec{B} = 0)$ is automatically satisfied.

(Note that $\vec{A}(\vec{x}, t) \to \vec{A}(\vec{x})$ in the static case.)

Now, plug $\vec{B} = \vec{\nabla} \times \vec{A}$ into $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$:

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{A})}_{\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}} = \mu_0 \vec{J}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

Gauge-invariance:
$$\begin{cases} \Phi \to \Phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t} \\ \vec{A} \to \vec{A} + \vec{\nabla} \Lambda \end{cases}$$

$\Rightarrow$ pick a gauge where $\boxed{\vec{\nabla} \cdot \vec{A} = 0}$ Coulomb gauge

$$\Rightarrow \quad \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

[$\vec{A}_{new} = \vec{A}_{old} + \vec{\nabla}\Lambda$ $\Rightarrow$ if we want

$\vec{\nabla}\cdot\vec{A}_{new} = 0 \Rightarrow \vec{\nabla}\cdot\vec{A}_{old} = -\nabla^2\Lambda$

$\Rightarrow \nabla^2\Lambda = -\vec{\nabla}\cdot\vec{A}_{old}$ $\Rightarrow$ can always solve this Poisson equation for $\Lambda(\vec{x}) \Rightarrow$ can choose Coulomb gauge.]

Let's solve $\nabla^2\vec{A} = -\mu_0\vec{J}$: this is just a Poisson equation $\Rightarrow$

$$\Rightarrow \boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}}$$

static vector potential in Coulomb gauge.

In a general gauge

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} + \vec{\nabla}\Lambda(\vec{x})$$

If you know $\vec{J}(\vec{x}) \Rightarrow$ can find $\vec{A} \Rightarrow$

$\Rightarrow$ know $\vec{B} = \vec{\nabla}\times\vec{A}$.

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \int d^3x' \; \vec{\nabla} \times \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$$\Rightarrow B^i = \frac{\mu_0}{4\pi} \int d^3x' \; \varepsilon^{ijk} \underbrace{\left(\nabla^j \frac{1}{|\vec{x} - \vec{x}'|} \right)}_{-\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}} J^k(\vec{x}')$$

$$\vec{B} = -\frac{\mu_0}{4\pi} \int d^3x' \; \frac{(\vec{x} - \vec{x}') \times \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi} \int d^3x' \; \frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}.}$$

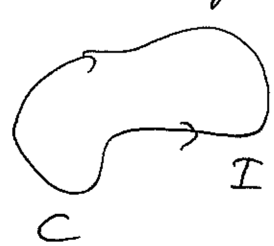
Biot & Savart Law

know $\vec{J} \Rightarrow$ find $\vec{B}$.

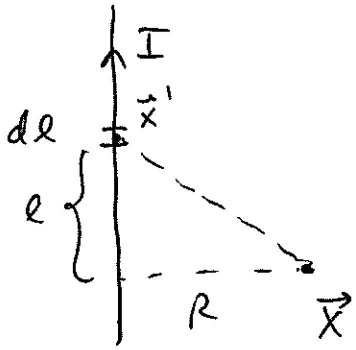
Note: Current $I \equiv \frac{\text{charge}}{\text{time}}$, such that

$$I \, d\vec{\ell} = \vec{J} \, d^3x$$

$\Rightarrow$ for a loop of current $\vec{B} = \frac{\mu_0}{4\pi} I \oint_C \frac{d\vec{\ell}' \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$



Example a straight wire carrying current I:



$$\Rightarrow B = \frac{\mu_0}{4\pi} I \int_{-\infty}^{\infty} \frac{dl}{l^2+R^2} \underbrace{\frac{R}{\sqrt{R^2+l^2}}}_{\text{sine of the angle between } d\vec{l} \text{ and } (\vec{x}-\vec{x}')}$$

$$\Rightarrow B = \frac{\mu_0}{4\pi} I R \int_{-\infty}^{\infty} \frac{dl}{\left[l^2+R^2\right]^{3/2}} = \frac{\mu_0}{4\pi} I R \left. \frac{l}{R^2\sqrt{R^2+l^2}} \right|_{-\infty}^{\infty}$$

$$\Rightarrow B = \frac{\mu_0}{2\pi} \frac{I}{R}.$$

Just FYI, all of this is even easier in relativistic notation: start with Maxwell eq's:

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu \qquad \text{(Gaussian units)}$$

$$\parallel$$

$$\partial_\nu \partial^\nu A^\mu - \partial_\nu \partial^\mu A^\nu$$

$\Rightarrow$ choose the Lorenz gauge: $\partial_\nu A^\nu = 0$

$\Rightarrow$ get $\square A^\mu = \frac{4\pi}{c} J^\mu$

In the case of statics, $\square = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \rightarrow -\vec{\nabla}^2$

$\Rightarrow$ get $\nabla^2 A^\mu(\vec{x}) = -\frac{4\pi}{c} J^\mu$

$\Rightarrow$ put $\mu = 0 \Rightarrow A^0(\vec{x}) = \Phi(\vec{x})$, $J^0 = c\rho(\vec{x})$

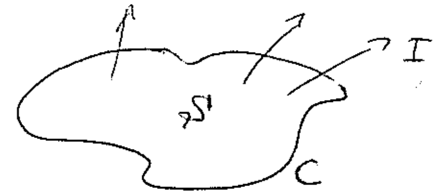
$\Rightarrow$ get $\boxed{\nabla^2 \Phi = -4\pi \rho}$ Poisson eq'n in Gaussian units

$\Rightarrow$ put $\mu = i \Rightarrow \boxed{\nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{J}}$ equation for $\vec{A}$ in Gaussian units

~ the analogue of $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ in SI units.

To derive an analogy of Gauss's law, integrate

$$\int_S da\ \hat{n}\cdot(\vec{\nabla}\times\vec{B}) = \oint_C \vec{B}\,d\vec{\ell} \quad \begin{pmatrix}\text{Stokes's}\\ \text{thm}\end{pmatrix}$$



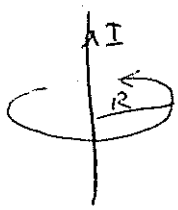
$$\parallel$$

$$\mu_0 \int_S da\ \hat{n}\cdot\vec{J} \Rightarrow \boxed{\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 \int_S da\ \hat{n}\cdot\vec{J} = \mu_0 I}$$

Ampere's law

I ~ total current through the loop.

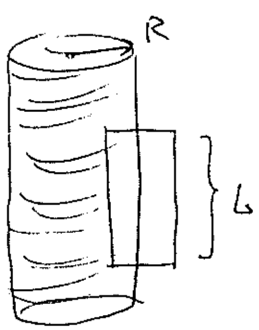
<u>Example</u>: find $\vec{B}$ of a straight wire carrying current I:



$$B\cdot 2\pi R = \mu_0 I \Rightarrow B = \frac{\mu_0}{2\pi}\frac{I}{R}$$

(cf. with what we found using Biot-Savart law earlier)

<u>Example</u>: infinite solenoid, N coils per unit length:



$$B_{in}\cdot L = \mu_0 I\cdot N\cdot L \Rightarrow B_{in} = \mu_0 I N$$

uniform magnetic field inside!

$B_{out} = 0$.

~~Finally, we have shown that~~ $\cancel{\boxed{\vec{\nabla}\cdot\vec{B} = 0}}$ ~~&~~ $\cancel{\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}}$

outside: $B_\varphi = 0$, $B_r = 0$

$\square \Rightarrow B(r_1) = 0$ ...

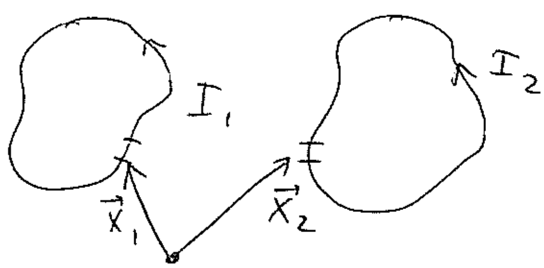
Ampere's Law.

The force on a current element $I_1 d\vec{\ell}_1$ due to magnetic field $\vec{B}$ is $\boxed{d\vec{F} = I_1\, d\vec{\ell}_1 \times \vec{B}}$

For a point charge q moving with velocity $\vec{v}$ write $\vec{F} = q\,\vec{v} \times \vec{B}$ (Lorentz force)

$\Rightarrow$ as $q\vec{v} \rightarrow I\,d\vec{\ell} \Rightarrow$ get

Imagine two loops of current: the force on loop #1 due to loop #2 is



$$\vec{F}_{12} = I_1 \int d\vec{\ell}_1 \times \vec{B}_2$$

Due to Biot & Savart law, $\vec{B}_2 = \frac{\mu_0}{4\pi} I_2 \int \frac{d\vec{\ell}_2 \times \vec{x}_{12}}{x_{12}^3}$

where $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$. Substituting:

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{\ell}_1 \times (d\vec{\ell}_2 \times \vec{x}_{12})}{x_{12}^3}$$

as $\frac{d\vec{\ell}_1 \times (d\vec{\ell}_2 \times \vec{x}_{12})}{x_{12}^3} = \frac{d\vec{\ell}_2\,(d\vec{\ell}_1 \cdot \vec{x}_{12})}{x_{12}^3} - \vec{x}_{12}\frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{x_{12}^3}$

and, since $\vec{\nabla}_1 \frac{1}{|\vec{x}_{12}|} = -\frac{\vec{x}_{12}}{|x_{12}|^3}$, the first term vanishes