Last time | **Magnetostatics (cont'd)**

$$\boxed{\begin{aligned}\vec{\nabla}\cdot\vec{B} &= 0\\ \vec{\nabla}\times\vec{B} &= \mu_0\vec{J}\end{aligned}}$$ differential equations of magnetostatics

used $\vec{B} = \vec{\nabla}\times\vec{A}$ and Coulomb gauge $(\vec{\nabla}\cdot\vec{A} = 0)$

to write

$$\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$$

$$\Rightarrow \boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}$$

Biot & Savart Law

relativistically, in Lorenz gauge $(\partial_\mu A^\mu = 0)$, Maxwell equ's become

$$\boxed{\Box A^\mu = \frac{4\pi}{c} J^\mu}$$

=> statics is

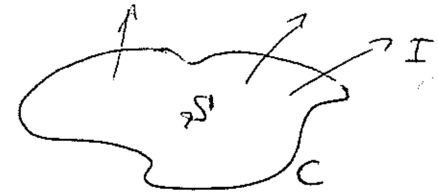
$$\nabla^2 A^\mu = -\frac{4\pi}{c} J^\mu$$ (Gauss units)

$\mu = 0$ => electrostatics

$\mu = i$ => magnetostatics

To derive an analogy of Gauss's law, integrate

$$\int_S da\ \hat{n}\cdot(\vec{\nabla}\times\vec{B}) = \oint_C \vec{B}\, d\vec{l} \quad \left(\text{Stokes's thm}\right)$$



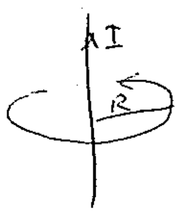
$$\| $$

$$\mu_0 \int_S da\ \hat{n}\cdot\vec{J} \Rightarrow \boxed{\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \int_S da\ \hat{n}\cdot\vec{J} = \mu_0 I}$$

Ampere's law

I ~ total current through the loop.

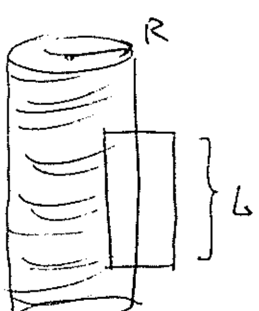
<u>Example</u>: find $\vec{B}$ of a straight wire carrying current I:



$$B\cdot 2\pi R = \mu_0 I \Rightarrow B = \frac{\mu_0}{2\pi}\frac{I}{R}$$

(cf. with what we found using Biot-Savart law earlier)

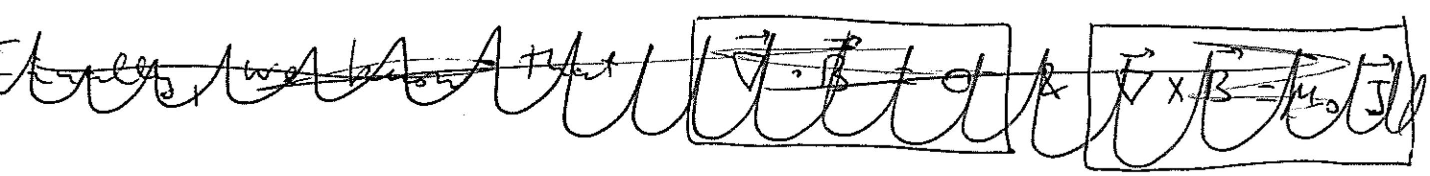
<u>Example</u>: infinite solenoid, N coils per unit length:



$$B_{in}\cdot L = \mu_0 I\cdot N\cdot L \Rightarrow B_{in} = \mu_0 I N$$

uniform magnetic field inside!

$B_{out} = 0$.



~~Finally, we know that~~ $\boxed{\cancel{\vec{\nabla}\cdot\vec{B} = 0}}$ ~~&~~ $\boxed{\cancel{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}}$

outside: $B_\varphi = 0$, $B_r = 0$

$\Rightarrow B(r) = 0$ ...

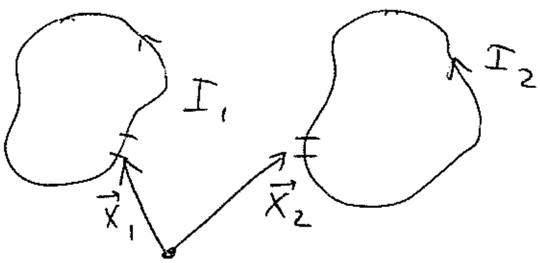
Forces: Ampere's Law

The force on a current element $I_1 d\vec{l}_1$ due to magnetic field $\vec{B}$ is $\boxed{d\vec{F} = I_1\, d\vec{l}_1 \times \vec{B}}$

For a point charge q moving with velocity $\vec{v}$ write $\vec{F} = q\,\vec{v} \times \vec{B}$ (Lorentz force)

$\Rightarrow$ as $q\vec{v} \rightarrow I\,d\vec{l} \Rightarrow$ get

Imagine two loops of current: the force on loop #1 due to loop #2 is



$$\vec{F}_{12} = I_1 \int d\vec{l}_1 \times \vec{B}_2$$

Due to Biot & Savart law, $\vec{B}_2 = \frac{\mu_0}{4\pi} I_2 \int \frac{d\vec{l}_2 \times \vec{x}_{12}}{x_{12}^3}$

where $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$. Substituting:

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12})}{x_{12}^3}$$

as $$\frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12})}{x_{12}^3} = \frac{d\vec{l}_2\,(d\vec{l}_1 \cdot \vec{x}_{12})}{x_{12}^3} - \vec{x}_{12}\,\frac{d\vec{l}_1 \cdot d\vec{l}_2}{x_{12}^3}$$

and, since $\vec{\nabla}_1 \frac{1}{|\vec{x}_{12}|} = -\frac{\vec{x}_{12}}{|x_{12}|^3}$, the first term vanishes and we write:

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{x}_{12}|^3} \vec{x}_{12}$$

attractive if $I_1 \parallel I_2$ repulsive if I_1 & I_2 anti-parallel

Ampere's law of force between two current loops.

As $I\, d\vec{l} = \vec{J}\, d^3x \Rightarrow$ for two localized current densities $\vec{F}_{12} = -\frac{\mu_0}{4\pi} \int d^3x_1\, d^3x_2 \frac{\vec{J}_1(\vec{x}_1) \cdot \vec{J}_2(\vec{x}_2)}{|\vec{x}_{12}|^3} \vec{x}_{12}$.

For current density Ampere's law gives:

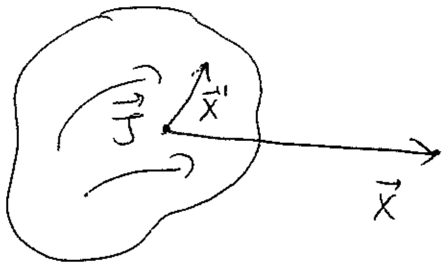
$$\boxed{\vec{F} = \int d^3x\, \vec{J}(\vec{x}) \times \vec{B}(\vec{x})}$$

The resulting torque is $\boxed{\vec{N} = \int d^3x\, \vec{x} \times (\vec{J}(\vec{x}) \times \vec{B}(\vec{x}))}$.

Magnetic Fields of a Localized Current Distribution:

Magnetic Dipole Moment.

Imagine a localized current distribution:



We need to find vector potential far away from the currents: start with

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$\Rightarrow$ to properly expand $\vec{A}(\vec{x})$ in powers of $\frac{1}{r}$ we need vector spherical harmonics ~ we'll maybe talk about them next quarter.

$\Rightarrow$ Instead expand

$$\frac{1}{|\vec{x}-\vec{x}'|} \simeq \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \ldots$$

$$\Rightarrow A_i(\vec{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|}\int d^3x' \, J_i(\vec{x}') + \frac{\mu_0}{4\pi}\frac{\vec{x}\,\cdot}{|\vec{x}|^3}\int d^3x' \,.$$

$$\cdot\,\vec{x}' \, J_i(\vec{x}') + \ldots$$

Now, $\int d^3x' \, J_i(\vec{x}') = \int d^3x' \left[\vec{\nabla}'\cdot\left(x_i' \, \vec{J}(\vec{x}')\right) - x_i' \, \vec{\nabla}\cdot\vec{J}\right\}$

First term becomes a surface integral

$$\int_S da\ x_i J_n = 0$$ as current is localized

Second term is also $\emptyset$ as $\vec{\nabla}\cdot\vec{J}=0$ (continuity)

$$\Rightarrow A_i(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{x}}{|\vec{x}|^3}\cdot\int d^3x'\ \vec{x}'\ J_i(\vec{x}')$$

as $\vec{\nabla}\cdot\vec{J}=0$.

Now, $0=\int d^3x'\ \vec{\nabla}'\cdot\left(x_i'x_j'\vec{J}(\vec{x}')\right) = \int d^3x'\left[x_i'J_j + x_j'J_i\right] \Rightarrow \int d^3x'\left[x_i'J_j + x_j'J_i\right]=0$

$$\Rightarrow \vec{x}\cdot\int d^3x'\ \vec{x}'\cdot J_i(\vec{x}') \equiv \sum_j x_j\int d^3x'\ x_j' J_i =$$

$$= -\frac{1}{2}\sum_j x_j\int d^3x'\left[x_i'J_j - x_j'J_i\right] =$$

$$= -\frac{1}{2}\sum_{j,k}\varepsilon_{ijk}\,x_j\int d^3x'\left(\vec{x}'\times\vec{J}\right)_k$$

as $\left(\vec{x}'\times\vec{J}\right)_k = \varepsilon_{ki'j'}\,x_{i'}'\,J_{j'}$ and

$$\varepsilon_{ijk}\,\varepsilon_{i'j'k} = \delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'}.$$

Finally we obtain $-\frac{1}{2}\left[\vec{x}\times\int d^3x'\left(\vec{x}'\times\vec{J}\right)\right]_i$, Such that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\left(-\frac{1}{2}\right)\frac{\vec{x}}{|\vec{x}|^3} \times \int d^3x' \, \vec{x}' \times \vec{J}$$

Definition. Defining magnetic dipole moment

$$\vec{m} = \frac{1}{2}\int d^3x' \; \vec{x}' \times \vec{J}(\vec{x}')$$

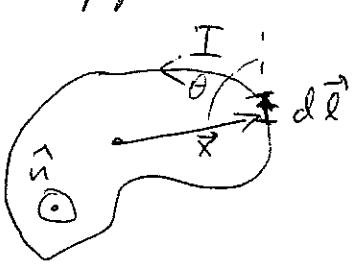
We obtain $\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}}$

$$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}}$$ (cf. with $\vec{E}$ of a dipole)

Definition $\vec{M} = \frac{1}{2}\vec{x}\times\vec{J}(\vec{x})$ is the magnetic moment density, or, magnetization.

(More precisely, $\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3} + \frac{8\pi}{3}\vec{m}\,\delta^3(\vec{x})\right]$)

Suppose the current is confined to a plane:



$$\vec{m} = \frac{1}{2} I \int \vec{x}\times d\vec{l}$$

$$\Rightarrow \text{as } |\vec{x}\times d\vec{l}|\cdot\frac{1}{2} = \frac{1}{2}\, x\, dl \cdot \sin\theta = \underbrace{da}_{\text{area element}}$$