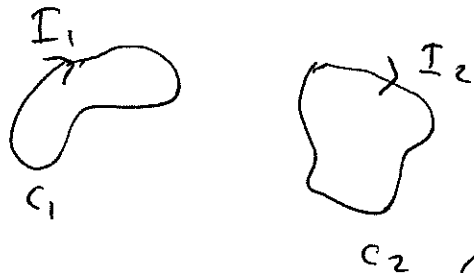
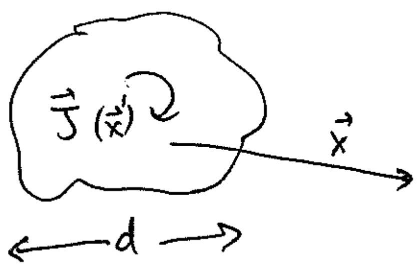
Last time | Forces: Ampere's Law



$$\boxed{\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1}\oint_{C_2} \frac{\vec{dl}_1 \cdot \vec{dl}_2}{|\vec{x}_{12}|^3}\, \vec{x}_{12}.}$$

$$\boxed{\vec{F} = \int d^3x\, \vec{J} \times \vec{B}}, \text{ torque } \boxed{\vec{N} = \int d^3x\, \vec{x} \times (\vec{J} \times \vec{B})}$$

Magnetic Field of a Localized Current Distribution: magnetic Dipole moment



$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$\Rightarrow$ we expanded $\vec{A}(\vec{x})$ in $\frac{d}{r}$

(d = size of localized current, r ~ distance to $\vec{x}$, $r = |\vec{x}|$)

$\Rightarrow$ the leading non-zero term was

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}}$$

where $\vec{m}$ is the magnetic dipole moment:

$$\boxed{\vec{m} \equiv \frac{1}{2} \int d^3x\, \vec{x} \times \vec{J}(\vec{x})}.$$

First term becomes a surface integral

$$\int_S da\ x_i J_n = 0 \quad \text{as current is localized}$$

Second term is also $\emptyset$ as $\vec{\nabla}\cdot\vec{J} = 0$ (continuity)

$$\Rightarrow A_i(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{x}}{|\vec{x}|^3}\cdot\int d^3x'\,\vec{x}'\,J_i(\vec{x}')$$

as $\vec{\nabla}\cdot\vec{J}=0$.

Now, $0 = \int d^3x'\,\vec{\nabla}'\cdot\left(x_i' x_j' \vec{J}(\vec{x}')\right) = \int d^3x' \left[x_i' J_j + \right.$

$\left. + x_j' J_i\right] \Rightarrow \int d^3x' \left[x_i' J_j + x_j' J_i\right] = 0$

$$\Rightarrow \vec{x}\cdot\int d^3x'\,\vec{x}'\cdot J_i(\vec{x}') \equiv \sum_j x_j \int d^3x'\, x_j' J_i =$$

$$= -\frac{1}{2}\sum_j x_j \int d^3x' \left[x_i' J_j - x_j' J_i\right] =$$

$$= -\frac{1}{2}\sum_{j,k} \varepsilon_{ijk}\, x_j \int d^3x' \left(\vec{x}'\times\vec{J}\right)_k$$

as $(\vec{x}'\times\vec{J})_k = \varepsilon_{ki'j'}\, x_{i'}' J_{j'}$ and

$$\varepsilon_{ijk}\varepsilon_{i'j'k} = \delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'}.$$

Finally we obtain $-\frac{1}{2}\left[\vec{x}\times\int d^3x'\,(\vec{x}'\times\vec{J})\right]_i$, Such that

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\left(-\frac{1}{2}\right)\frac{\vec{X}}{|\vec{X}|^3} \times \int d^3x' \, \vec{x}' \times \vec{J}$$

Definition. Defining magnetic dipole moment

$$\vec{m} = \frac{1}{2}\int d^3x' \; \vec{x}' \times \vec{J}(\vec{x}')$$

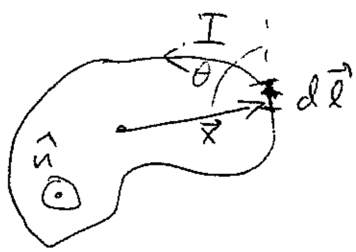
We obtain
$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}}$$ (cf. with $\vec{E}$ of a dipole)

Definition $\vec{M} = \frac{1}{2}\vec{x}\times\vec{J}(\vec{x})$ is the magnetic moment density, or, magnetization.

(More precisely, $\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3} + \frac{8\pi}{3}\vec{m}\,\delta^3(\vec{x})\right]$),

Suppose the current is confined to a plane:



$$\vec{m} = \frac{1}{2} I \int \vec{x}\times d\vec{l}$$

$\Rightarrow$ as $|\vec{x}\times d\vec{l}|\cdot\frac{1}{2} = \frac{1}{2}\, x\, dl \cdot \sin\theta = \underbrace{da}_{\text{area element}}$

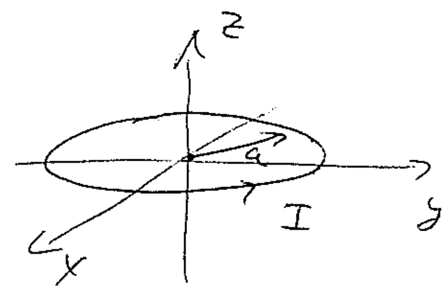
$\Rightarrow \left| \frac{1}{2} \int \vec{x} \times d\vec{l} \right| = S$ (area of the loop)

$\Rightarrow |\vec{m}| = I \cdot S$, or $\boxed{\vec{m} = I S \hat{n}}$

$\hat{n}$ is pointing out of the plane

$\vec{m}$ is independent of origin. Can you prove that?

<u>Example</u>: current loop:

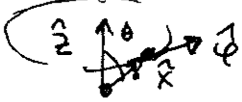


$\Rightarrow \vec{m} = I \cdot \pi a^2 \cdot \hat{n} = I \pi a^2 \hat{z}$.

far from the loop.

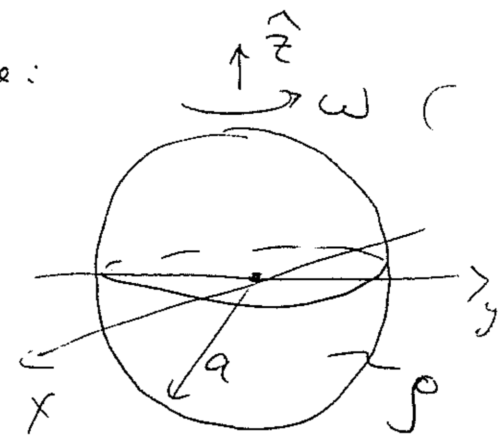
$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} = \frac{\mu_0 I a^2}{4} \cdot \frac{\hat{z} \times \vec{x}}{\vec{x}^3} \Rightarrow \text{in spherical coordinates}$$

(as $\hat{z} \times \hat{x} = \hat{\varphi} \sin\theta$)



$$A_\varphi = \frac{\mu_0 I a^2}{4} \frac{\sin\theta}{r^2} \qquad A_\theta = A_r = 0.$$

<u>Example</u>: find magnetic dipole moment of a rotating uniformly charged sphere:



$$\vec{m} = \frac{1}{2}\int d^3x \quad \vec{x} \times \vec{J}$$

$$\vec{J} = \rho \cdot \vec{V} = \rho\, \vec{\omega} \times \vec{x}$$

$$\Rightarrow \vec{m} = \frac{\rho}{2}\int d^3x \quad \vec{x} \times (\vec{\omega} \times \vec{x}) =$$

$$= \frac{\rho}{2}\int d^3x \left[\vec{\omega}\, |\vec{x}|^2 - \vec{x}\cdot(\vec{x}\cdot\vec{\omega})\right]$$

$$\Rightarrow \text{as } \vec{\omega} = \omega\,\hat{z} \Rightarrow m_x = m_y = 0$$

$$\Rightarrow m_z = \frac{\rho}{2}\omega \int d^3x \left[r^2 - z^2\right] =$$

$$= \frac{\rho}{2}\omega \cdot 2\pi \int_0^a dr \cdot r^2 \int_{-1}^{1} d\cos\theta \left[r^2 - r^2\cos^2\theta\right] =$$

$$= \frac{\rho}{\cancel{2}}\omega \cdot \cancel{2}\pi\, \frac{a^5}{5}\left[2 - \frac{2}{3}\right] = \pi\,\omega\,\rho\, a^5 \cdot \frac{4}{15}$$

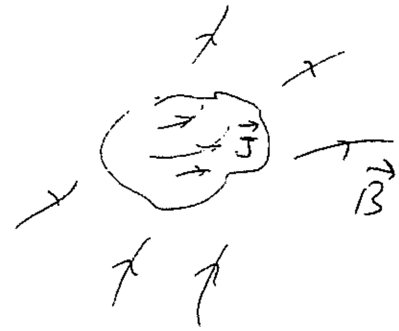
$$\Rightarrow \text{as } q = \frac{4}{3}\pi a^3 \rho \Rightarrow \boxed{m = \frac{1}{5} q\,\omega\, a^2}.$$

<u>Torque</u> on $\vec{m}$:

$$\boxed{\vec{N} = \vec{m} \times \vec{B}(0)}.$$

Force and Energy of a Localized Current.

Consider a system of localized currents in external magnetic induction $\vec{B}$:



$$\vec{F} = \int d^3x \; \vec{J}(\vec{x}) \times \vec{B}(\vec{x})$$

If $\vec{B}$ is slowly varying, write

$$\vec{B}(\vec{x}) = \vec{B}(0) + (\vec{x} \cdot \vec{\nabla})\, \vec{B}(0) + \ldots$$

$$\Rightarrow \vec{F} = \underbrace{\left[\int d^3x \; \vec{J}(\vec{x}) \right]}_{\to 0} \times \vec{B}(0) + \int d^3x \; \vec{J}(\vec{x}) \times (\vec{x} \cdot \vec{\nabla})\, \vec{B}(0)$$

$$\Rightarrow F_i = \int d^3x \; \varepsilon_{ijk} \, J_j(\vec{x}) \cdot (\vec{x} \cdot \vec{\nabla})\, B_k(0) =$$

$$= \int d^3x \; \varepsilon_{ijk} \, J_j(\vec{x}) \, x_l \, (\partial_l B_k)\big|_{\vec{x}=0} =$$

$$= (\partial_l B_k)\big|_{\vec{x}=0} \; \varepsilon_{ijk} \int d^3x \; x_l J_j$$

$$\Rightarrow \text{as} \int d^3x \, (x_i J_j + x_j J_i) = 0 \quad \Rightarrow$$

$$F_i = (\partial_l B_k)\big|_{\vec{x}=0} \; \varepsilon_{ijk} \underbrace{\frac{1}{2} \int d^3x \left[x_l J_j - x_j J_l \right]}_{\varepsilon_{ljn} \cdot m_n}$$

(as $m_i = \frac{1}{2}\varepsilon_{ijk}\int d^3x\, x_j J_k$ $\Rightarrow$

$$\Rightarrow \varepsilon_{ljn} \cdot m_n = \frac{1}{2} \underbrace{\varepsilon_{ljn}\varepsilon_{nj'k}}_{\delta_{lj'}\delta_{jk} - \delta_{lk}\delta_{jj'}} \int d^3x\, x_{j'} J_k =$$

$$= \frac{1}{2}\int d^3x \left[x_l J_j - x_j J_l\right] \Big)$$

$$\Rightarrow F_i = (\partial_l B_k)\big|_{\vec{x}=0}\, \varepsilon_{ijk}\,\varepsilon_{ljn}\, m_n =$$

$$= (\partial_l B_k)\big|_{\vec{x}=0}\, m_n \cdot \left[\delta_{il}\delta_{kn} - \delta_{in}\delta_{kl}\right] =$$

$$= m_k (\partial_i B_k)\big|_{\vec{x}=0} - m_i \underbrace{(\partial_k B_k)\big|_{\vec{x}=0}}_{= 0 \text{ as } \vec{\nabla}\cdot\vec{B} = 0}$$

$$\Rightarrow \boxed{\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})}$$

If $\vec{F} = -\vec{\nabla} U$, with U the potential energy, $\Rightarrow$ $\boxed{U = -\vec{m}\cdot\vec{B}}$

tends to align dipoles with the magnetic induction $\vec{B}$.