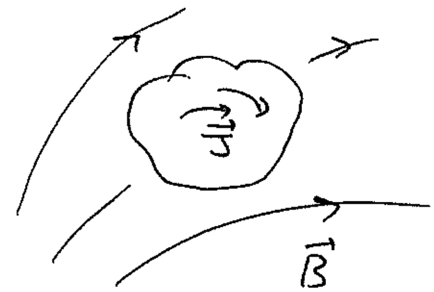
Last time | Worked out a few examples on the magnetic dipole moment: $\vec{m} = \frac{1}{2}\int d^3x\ \vec{x} \times \vec{J}$

Force on a localized current:

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = -\vec{\nabla} U$$

$$U = -\vec{m}\cdot\vec{B}$$

↑ energy



In general one can show that the following multipole expansion is valid for $\vec{A}$:

$$\boxed{\vec{A}(\vec{x}) = \mu_0 \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{1}{i\ell} M_{\ell m} \vec{L}\left(\frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}}\right)}$$

where

$$\vec{L} \equiv -i\,\vec{r} \times \vec{\nabla}$$ is the orbital angular momentum operator,

and

$$M_{\ell m} \equiv i \sqrt{\frac{\ell}{\ell+1}} \int d^3x \; r^{\ell}\, \vec{X}^{*}_{\ell m}(\theta,\varphi) \cdot \vec{J}(\vec{x})$$

are the magnetic multipole moments.

Here, $\vec{X}_{\ell m}(\theta,\varphi) \equiv \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L}\, Y_{\ell m}(\theta,\varphi)$ is the vector spherical harmonic.

(see Zangwill, Sec. 11.4.2–4; Jackson 9.7).

Macroscopic Equations of Magnetostatics

Similar to electrostatics, let's divide the currents into "free" and "bound".

Bound currents are due to magnetic moments of molecules/atoms in the medium and are described by magnetization (magnetic moment per unit volume) $\vec{M}(\vec{x}) = \sum_i N_i \langle \vec{m}_i \rangle$

magn. dip. moment of type i molecule

molecules of type i per unit volume

The resulting vector-potential is

$\vec{M}(\vec{x}) = \sum_n \vec{m}_i\, \delta(\vec{x} - \vec{x}_n)$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \left\{ \underbrace{\frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}}_{\text{free current}} + \underbrace{\frac{\vec{M}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}}_{\text{magnetization/ bound currents}} \right\}$$

The 2nd term is

$$\int d^3x' \frac{\vec{M}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = \int d^3x' \, \vec{M}(\vec{x}') \times \nabla' \frac{1}{|\vec{x} - \vec{x}'|} =$$

$$= -\vec{\nabla}' \times \left[\vec{M} \frac{1}{|\vec{x} - \vec{x}'|}\right] + (\vec{\nabla}' \times \vec{M}) \frac{1}{|\vec{x} - \vec{x}'|}$$

$$= (\text{parts}) = \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} \vec{\nabla}' \times \vec{M}(\vec{x}'),$$ such that

$$\left[\int_V d^3x \, \vec{\nabla} \times \vec{A} = \int_S da \, \hat{n} \times \vec{A}\right]$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}') + \vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$\Rightarrow$ effective current density due to $\vec{M}$ is

$$\vec{J}_M = \vec{\nabla} \times \vec{M}$$

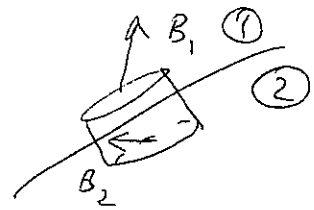
$$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \vec{\nabla} \times \vec{M} \right]}$$

Define magnetic field $\boxed{\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}}$:

$\Rightarrow$ Macroscopic equations of magnetostatics are $\boxed{\vec{\nabla} \times \vec{H} = \vec{J}}$ and $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$

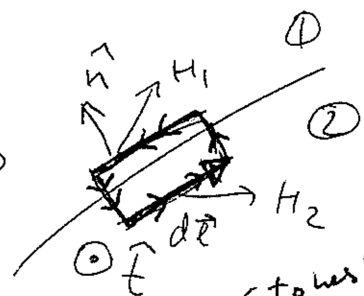
(cf. $\vec{\nabla} \cdot \vec{D} = \rho_f$, $\vec{\nabla} \times \vec{E} = 0$ in electrostatics)

Boundary conditions: $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$



$$\boxed{B_{1n} = B_{2n}}$$

$\vec{\nabla} \times \vec{H} = \vec{J} \Rightarrow$

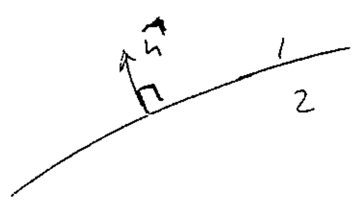


$\Rightarrow$

stokes's th'm

$(\vec{H}_2 - \vec{H}_1) \cdot d\vec{l} = \vec{K} \cdot \hat{t}\, dl$, where $\vec{K}$ = surface current density (amps/m²).

$$\Rightarrow \boxed{\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}}$$

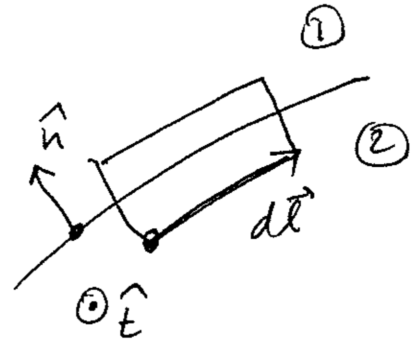


see next page

At the boundary of two media with magnetizations $\vec{M}_1$ & $\vec{M}_2$: $\vec{J}_M = \vec{\nabla} \times \vec{M}$ and $\hat{n} \times (\vec{H}_1 - \vec{H}_2) =$

$$= \hat{n} \times \left[\frac{1}{\mu_0} \vec{B}_1 - \vec{M}_1 - \frac{1}{\mu_0} \vec{B}_2 + \vec{M}_2 \right] = \vec{K}$$

as $\hat{t} = \dfrac{d\vec{l} \times \hat{n}}{dl}$



$\Rightarrow \hat{t}\, dl = d\vec{l} \times \hat{n}$

$\Rightarrow (\vec{H}_2 - \vec{H}_1) \cdot d\vec{l} = \vec{K} \cdot \hat{t}\, dl$

$\Rightarrow (\vec{H}_2 - \vec{H}_1) \cdot d\vec{l} = \vec{K} \cdot (d\vec{l} \times \hat{n}) = d\vec{l} \cdot (\hat{n} \times \vec{K})$

as this is true for any $d\vec{l}$ along the boundary,

$$(\vec{H}_2 - \vec{H}_1)_t = \hat{n} \times \vec{K}$$

$$\Rightarrow \hat{n} \times (\vec{H}_2 - \vec{H}_1)_t = \hat{n} \times (\vec{H}_2 - \vec{H}_1)$$

$$\parallel$$

$$\hat{n} \times (\hat{n} \times \vec{K}) = \hat{n} \underbrace{(\hat{n} \cdot \vec{K})}_{=0 \text{ as } \hat{n} \perp \vec{K}} - \vec{K}$$

$$\Rightarrow \hat{n} \times (\vec{H}_2 - \vec{H}_1) = -\vec{K}$$, or

$$\boxed{\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}}$$ as desired

Therefore $\frac{1}{\mu_0} \hat{n} \times (\vec{B}_1 - \vec{B}_2) = \vec{K} + \vec{K}_M$

where $\vec{K}_M = \hat{n} \times (\vec{M}_1 - \vec{M}_2)$ ~ surface current due to magnetization.

What's missing? Relation between $\vec{B}$ & $\vec{H}$!

For linear isotropic homogeneous media

$$\boxed{\vec{B} = \mu \vec{H}}$$, μ is magnetic permeability.

As $\vec{B} = \mu_0 (\vec{H} + \vec{M}) \Rightarrow \vec{M} = \frac{1}{\mu_0} \vec{B} - \vec{H} = \left(\frac{\mu}{\mu_0} - 1\right) \vec{H} =$

$= \chi_m \vec{H} \Rightarrow \vec{M} = \chi_m \vec{H}$, $\chi_m = \frac{\mu}{\mu_0} - 1$

magnetic susceptibility

$\chi_m > 0$ $(\mu > \mu_0)$ ~ paramagnetic (atoms/molecules have some angular momentum)

$\chi_m < 0$ $(\mu < \mu_0)$ ~ diamagnetic (no net angular momentum on atoms/molecules ~ resist magnetic field)

$\vec{M} \neq 0$ "independent" of $\vec{H}$ (even for $\vec{H} = 0$) ~ ferromagnetic

(paramagnetic materials with self-interactions between spins ~ e.g. Ising model)

(hard magnetic materials)

Solving Boundary-Value Problems in Magnetostatics

(I) No ferromagnetics (no frozen $\vec{M} \neq 0$ for all $\vec{H}$)

A. Vector Potential ($\vec{J} \neq 0$)

$\vec{B} = \vec{\nabla} \times \vec{A}$ => in Coulomb gauge $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

=> given $\vec{J}$ can always find $\vec{A}$. (vacuum)

As $\vec{\nabla} \times \vec{H} = \vec{J} = \vec{\nabla} \times \left(\frac{\vec{B}}{\mu}\right) = -\frac{1}{\mu} \nabla^2 \vec{A}$ =>

in medium $\nabla^2 \vec{A} = -\mu \vec{J}$.

B. $\vec{J} = 0$ => Magnetic Scalar Potential.

$\vec{J} = 0 \Rightarrow \vec{\nabla} \times \vec{H} = 0$, $\vec{\nabla} \cdot \vec{B} = 0$

=> write $\vec{H} = -\vec{\nabla} \Phi_M \Rightarrow \vec{B} = \mu \vec{H} \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$

gives $\vec{\nabla} \cdot \vec{H} = 0 \Rightarrow \nabla^2 \Phi_M = 0$ ~ Laplace eqn.

(non-zero $\vec{H}$, $\vec{B}$ may be due to boundary conditions)

(II) Ferromagnetics ($\vec{M} \neq 0$, $\vec{J} = 0$)

A. Vector potential.

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ => as $\vec{\nabla} \times \vec{H} = \vec{J}$ =>

$\Rightarrow \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M}\right) = \vec{J} \Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{\nabla} \times \vec{M} - \mu_0 \vec{J}$