Last time | Macroscopic Equations of Magnetostatics (cont'd)

Def. Magnetization $\vec{M} = \dfrac{\text{magnetic dipole moment}}{\text{volume}}$

Solution of Poisson equation for $\vec{A}(\vec{x})$ in unlimited space:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}\,') + \vec{\nabla}' \times \vec{M}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}$$

$\Rightarrow$ effective current density due to $\vec{M}$ is

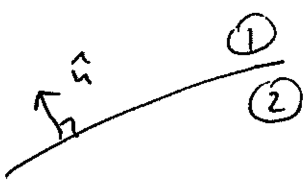
$$\vec{J}_M = \vec{\nabla} \times \vec{M}.$$

Def. Magnetic field $\boxed{\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}}$

$\Rightarrow$ Maxwell equations for magnetostatics are

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}} \quad \text{and} \quad \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

Boundary matching:



$$\boxed{B_{1n} = B_{2n}}$$

$$\boxed{\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}} \sim \text{surface current density}$$

$$\boxed{\vec{K}_M = \hat{n} \times (\vec{M}_1 - \vec{M}_2)} \quad \text{surface current density due to magnetization}$$

Therefore $\frac{1}{\mu_0}\hat{n}\times(\vec{B}_1-\vec{B}_2)=\vec{K}+\vec{K}_M$ (205)

where $\vec{K}_M=\hat{n}\times(\vec{M}_1-\vec{M}_2)$ ~ surface current due to magnetization.

<u>What's missing?</u> <u>Relation between $\vec{B}$ & $\vec{H}$!</u>

For linear isotropic homogeneous media

$$\boxed{\vec{B}=\mu\vec{H}}\quad,\ \mu \text{ is magnetic permeability.}$$

As $\vec{B}=\mu_0(\vec{H}+\vec{M}) \Rightarrow \vec{M}=\frac{1}{\mu_0}\vec{B}-\vec{H}=\left(\frac{\mu}{\mu_0}-1\right)\vec{H}=$

$=\chi_m\vec{H} \Rightarrow \vec{M}=\chi_m\vec{H}$, $\chi_m=\frac{\mu}{\mu_0}-1$

magnetic susceptibility

$\chi_m>0$ $(\mu>\mu_0)$ ~ paramagnetic (atoms/molecules have some angular momentum)

$\chi_m<0$ $(\mu<\mu_0)$ ~ diamagnetic (no net angular momentum on atoms/molecules ~ resist magnetic field)

$\vec{M}\neq 0$ "independent" of $\vec{H}$ (even for $\vec{H}=0$) ~ ferromagnetic

(paramagnetic materials with self-interactions between spins ~ e.g. Ising model)

(hard magnetic materials)

Superconductors: $\vec{B}=0 \Rightarrow \vec{M}=-\vec{H} \Rightarrow \chi_m=-1 \Rightarrow$

$\Rightarrow \mu=0$ "perfect diamagnets"

Solving Boundary-Value Problems in Magnetostatics.

(I) No ferromagnetics (no frozen $\vec{M} \neq 0$ for all $\vec{H}$)

A. Vector Potential ($\vec{J} \neq 0$)

$\vec{B} = \vec{\nabla} \times \vec{A}$ => in Coulomb gauge $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

=> given $\vec{J}$ can always find $\vec{A}$. (vacuum)

As $\vec{\nabla} \times \vec{H} = \vec{J} = \vec{\nabla} \times \left(\frac{\vec{B}}{\mu}\right) = -\frac{1}{\mu} \nabla^2 \vec{A}$ =>

in medium $\nabla^2 \vec{A} = -\mu \vec{J}$.

B. $\vec{J} = 0$ => Magnetic Scalar Potential.

$\vec{J} = 0 \Rightarrow \vec{\nabla} \times \vec{H} = 0$, $\vec{\nabla} \cdot \vec{B} = 0$

=> write $\vec{H} = -\vec{\nabla} \Phi_M \Rightarrow \vec{B} = \mu \vec{H} \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$

gives $\vec{\nabla} \cdot \vec{H} = 0 \Rightarrow \nabla^2 \Phi_M = 0$ ~ Laplace equ.

(non-zero $\vec{H}$, $\vec{B}$ may be due to boundary conditions)

(II) Ferromagnetics ($\vec{M} \neq 0$ when $\vec{H} = 0$)

A. Vector potential.

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ => as $\vec{\nabla} \times \vec{H} = \vec{J}$ =>

$\Rightarrow \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M}\right) = \vec{J} \Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{\nabla} \times \vec{M} - \mu_0 \vec{J}$

⇒ without "special" treatment of boundaries:

$$\boxed{\vec A = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec J + \vec\nabla' \times \vec M(\vec x')}{|\vec x - \vec x'|}}$$

← if $\vec J \neq 0$ just add $\vec J$ in the numerator

with "special" treatment of boundaries:

$$\boxed{\vec A(\vec x) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec J + \vec\nabla' \times \vec M(\vec x')}{|\vec x - \vec x'|} + \frac{\mu_0}{4\pi}\oint_S da' \frac{\vec K + \vec M(\vec x') \times \hat n'}{|\vec x - \vec x'|}}$$

(Remember ~ $\vec K = (\vec M_2 - \vec M_1) \times \hat n$ is the surface current)

B. Magnetic Scalar Potential

(if $\vec J = 0$)
$\Rightarrow \vec\nabla \times \vec H = 0$

$0 = \vec\nabla \cdot \vec B = \mu_0 \vec\nabla \cdot (\vec H + \vec M) = 0 \Rightarrow$ defining $\vec H = -\vec\nabla \Phi_M$

we get $\nabla^2 \Phi_M = \vec\nabla \cdot \vec M$ ~ Poisson-like eqn.

$$\Rightarrow \boxed{\Phi_M(\vec x) = -\frac{1}{4\pi}\int d^3x' \frac{\vec\nabla' \cdot \vec M(\vec x')}{|\vec x - \vec x'|}}$$

no explicit boundaries term (boundaries are included in careful eval. of $\vec\nabla \cdot \vec M$)

With explicit boundaries' term

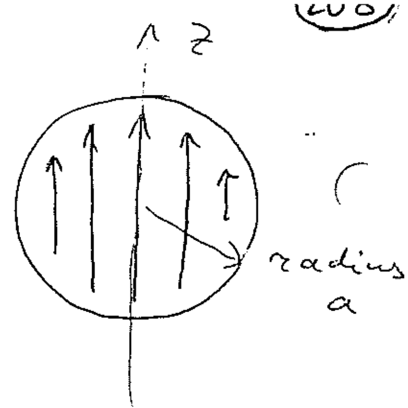
$$\boxed{\Phi_M(\vec x) = -\frac{1}{4\pi}\int d^3x' \frac{\vec\nabla' \cdot \vec M(\vec x')}{|\vec x - \vec x'|} + \frac{1}{4\pi}\oint_S da' \frac{\hat n' \cdot \vec M(\vec x')}{|\vec x - \vec x'|}}$$

"magnetic surface-charge" density

$$\boxed{\sigma_M = \hat n \cdot \vec M.}$$

Example: Ferromagnetic sphere:

$\vec{M}$ constant, $\vec{J}=0$, $\vec{M}=M\cdot\hat{z}$



Find $\vec{B}$, $\vec{H}$ everywhere.

$\Rightarrow$ Use magnetic potential:

$$\vec{\nabla}\cdot\vec{M}=0 \Rightarrow \Phi_M = \frac{1}{4\pi}\oint_S da' \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} =$$

$$= \frac{M}{4\pi}\cdot a^2\cdot\int d\Omega' \frac{\cos\theta'}{|\vec{x}-\vec{x}'|}$$

Using $\displaystyle \frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$

and $Y_{10}(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta$ we get

$$\Phi_M = \frac{Ma^2}{\cancel{4\pi}}\int_0^{2\pi}d\varphi'\int_{-1}^{1}d\cos\theta'\,\cancel{4\pi}\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\cdot$$

$$\cdot Y_{lm}^*(\theta',\varphi')\cdot\sqrt{\frac{4\pi}{3}}\,Y_{10}(\theta',\varphi')\,Y_{lm}(\theta,\varphi) = \begin{pmatrix}\text{only } l=1\\ m=0 \text{ is} \neq 0\end{pmatrix}$$

$$= Ma^2\,\frac{1}{3}\,\frac{r_<}{r_>^2}\cdot\cos\theta, \text{ where } r_{<\atop>} = {\min\atop\max}\{r,a\}$$

$\Rightarrow$ inside $\quad \Phi_M = \frac{1}{3}Mr\cos\theta = \frac{1}{3}Mz$

outside $\quad \Phi_M = \frac{1}{3}\frac{Ma^3}{r^2}\cos\theta$

Table: Boundary-Value Problems in Magnetostatics

method	no Ferromagnetics $\vec{B}=\mu\vec{H}$	Ferromagnetics ($\vec{M}\neq 0$ for $\vec{H}=0$)
$\vec{A}$	$\nabla^2\vec{A}=-\mu\vec{J}$ $\Rightarrow \vec{A}(\vec{x})=\frac{\mu}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}')}{\|\vec{x}-\vec{x}'\|}$ (∞ space)	$\nabla^2\vec{A}=-\mu_0[\vec{J}+\vec{\nabla}\times\vec{M}]$ $\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int_V d^3x' \frac{\vec{J}(\vec{x}')+\vec{\nabla}'\times\vec{M}(\vec{x}')}{\|\vec{x}-\vec{x}'\|}$ $+\frac{\mu_0}{4\pi}\int_S da' \frac{\vec{K}(\vec{x}')+\vec{M}(\vec{x}')\times\hat{n}'}{\|\vec{x}-\vec{x}'\|}$ surface term
Φ_M (only for $\vec{J}=0$)	$\nabla^2\Phi_M=0$ Solve Laplace eq'n with b.c.'s.	$\nabla^2\Phi_M=\vec{\nabla}\cdot\vec{M}$ $\Rightarrow \Phi_M(\vec{x})=-\frac{1}{4\pi}\int_V d^3x' \frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{\|\vec{x}-\vec{x}'\|}$ $+\frac{1}{4\pi}\int_S da' \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{\|\vec{x}-\vec{x}'\|}$