Table: Boundary-Value Problems in Magnetostatics

method	no Ferromagnetics $\vec{B} = \mu \vec{H}$	Ferromagnetics ($\vec{M} \neq 0$ for $\vec{H} = 0$)
$\vec{A}$	$\nabla^2 \vec{A} = -\mu \vec{J}$ $\Rightarrow \vec{A}(\vec{x}) = \frac{\mu}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{\lvert \vec{x} - \vec{x}' \rvert}$ (∞ space)	$\nabla^2 \vec{A} = -\mu_0 [\vec{J} + \vec{\nabla} \times \vec{M}]$ $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \frac{\vec{J}(\vec{x}') + \vec{\nabla}' \times \vec{M}(\vec{x}')}{\lvert \vec{x} - \vec{x}' \rvert}$ $+ \frac{\mu_0}{4\pi} \int_S da' \frac{\vec{K}(\vec{x}') + \vec{M}(\vec{x}') \times \hat{n}'}{\lvert \vec{x} - \vec{x}' \rvert}$ surface term
Φ_M (only for $\vec{J} = 0$)	$\nabla^2 \Phi_M = 0$ Solve Laplace eq'n with b.c.'s.	$\nabla^2 \Phi_M = \vec{\nabla} \cdot \vec{M}$ $\Rightarrow \Phi_M(\vec{x}) = -\frac{1}{4\pi} \int_V d^3x' \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{\lvert \vec{x} - \vec{x}' \rvert}$ $+ \frac{1}{4\pi} \int_S da' \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{\lvert \vec{x} - \vec{x}' \rvert}$

$\Rightarrow$ without "special" treatment of boundaries:

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}+\vec{\nabla}'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

$\Leftarrow$ if $\vec{J}\neq 0$ just add $\vec{J}$ in the numerator

with "special" treatment of boundaries:

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}+\vec{\nabla}'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\mu_0}{4\pi}\oint_S da' \, \frac{\vec{K}+\vec{M}(\vec{x}')\times\hat{n}'}{|\vec{x}-\vec{x}'|}}$$

(Remember ~ $\vec{K} = (\vec{M}_2 - \vec{M}_1)\times\hat{n}$ is the surface current)

B. Magnetic Scalar Potential (if $\vec{J}=0$)

$\Rightarrow \vec{\nabla}\times\vec{H}=0$

$0 = \vec{\nabla}\cdot\vec{B} = \mu_0\vec{\nabla}\cdot(\vec{H}+\vec{M}) = 0 \Rightarrow$ defining $\vec{H} = -\vec{\nabla}\,\Phi_M$

we get $\nabla^2\Phi_M = \vec{\nabla}\cdot\vec{M}$ ~ Poisson-like eqn.

$$\Rightarrow \boxed{\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \, \frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

no explicit boundaries term (boundaries are included in careful eval. of $\vec{\nabla}\cdot\vec{M}$)

With explicit boundaries' term

$$\boxed{\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \, \frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{1}{4\pi}\oint_S da' \, \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

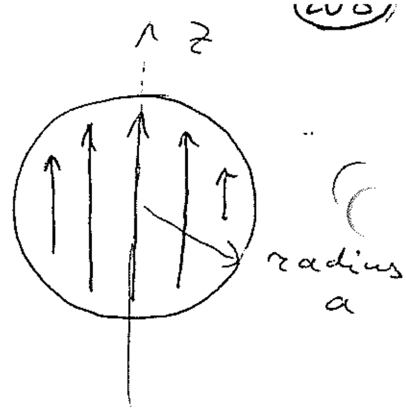
"magnetic surface-charge" density

$$\boxed{\sigma_M = \hat{n}\cdot\vec{M}.}$$

Example: Ferromagnetic sphere:

$\vec{M}$ constant, $\vec{J} = 0$, $\vec{M} = M \cdot \hat{z}$

Find $\vec{B}$, $\vec{H}$ everywhere.



$\Rightarrow$ Use magnetic potential:

$$\vec{\nabla} \cdot \vec{M} = 0 \Rightarrow \Phi_M = \frac{1}{4\pi} \oint_S da' \, \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} =$$

$$= \frac{M}{4\pi} \cdot a^2 \cdot \int d\Omega' \, \frac{\cos\theta'}{|\vec{x} - \vec{x}'|}.$$

Using $\displaystyle \frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$

and $\displaystyle Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$ we get

$$\Phi_M = \frac{Ma^2}{\cancel{4\pi}} \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\cos\theta' \, \cancel{4\pi} \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \cdot$$

$$\cdot Y_{lm}^*(\theta', \varphi') \cdot \sqrt{\frac{4\pi}{3}} Y_{10}(\theta', \varphi') Y_{lm}(\theta, \varphi) = \left(\begin{array}{c}\text{only } l=1 \\ m=0 \text{ is} \neq 0\end{array}\right)$$

$$= Ma^2 \frac{1}{3} \frac{r_<}{r_>^2} \cdot \cos\theta, \text{ where } r_{\lessgtr} = \begin{array}{c}\min \\ \max\end{array} \{ r, a \}$$

$\Rightarrow$ inside $\displaystyle \Phi_M = \frac{1}{3} M r \cos\theta = \frac{1}{3} M z$

outside $\displaystyle \Phi_M = \frac{1}{3} \frac{Ma^3}{r^2} \cos\theta$

$\Rightarrow \vec{H} = -\vec{\nabla}\Phi_M \Rightarrow \boxed{\vec{H}_{in} = -\frac{1}{3}\vec{M}}$ inside

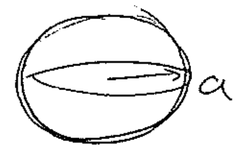
$\vec{B} = \mu_0(\vec{H} + \vec{M}) \Rightarrow \boxed{\vec{B}_{in} = \frac{2}{3}\vec{M}\mu_0}$ inside.

Outside : $\vec{H} = -\vec{\nabla}\left(\frac{1}{3}\frac{Ma^3}{r^2}\cos\theta\right)$; $\vec{B} = \mu_0\vec{H}$.

$\Rightarrow \vec{B} = -\frac{1}{3}\mu_0 M a^3\,\vec{\nabla}\left(\frac{\hat{z}\cdot\vec{r}}{r^3}\right) \Rightarrow$ dipole moment

$\frac{\hat{z}}{r^3} - \frac{3(\hat{z}\cdot\vec{r})\vec{r}}{r^5}$ makes sense $\vec{m} = \frac{4\pi a^3}{3}\vec{M}$.

Example: a sphere with permeable magnetic material $(\vec{B} = \mu\vec{H}$ inside, $\vec{B} = \mu_0\vec{H}$ outside$)$ in external magnetic field $\vec{H}_0$ $\uparrow\ \uparrow\ \uparrow \vec{H}_0$



$\nabla^2\Phi_M = 0$ inside & outside

$\Rightarrow \Phi_{inside} = \sum_{\ell} A_\ell r^\ell P_\ell(\cos\theta)$, $r<a$

$\Phi_{outside} = -H_0 r P_1(\cos\theta) + \sum_{\ell=1}^{\infty}\frac{B_\ell}{r^{\ell+1}}P_\ell(\cos\theta)$, $r>a$

as $H_{in,t} = H_{out,t} \Rightarrow$

$\Rightarrow$ (1) $\frac{\partial\Phi_1}{\partial\theta} = \frac{\partial\Phi_2}{\partial\theta}$ at $r=a$; $B_{1n} = B_{2n} \Rightarrow \mu H_{in,n} = \mu_0 H_{out,n}$

or just continuity, $\Phi_1 = \Phi_2$ at $r=a$

(2) $\mu\left.\frac{\partial\Phi_{in}}{\partial r}\right|_{r=a} = \mu_0\left.\frac{\partial\Phi_{out}}{\partial r}\right|_{r=a}$

$P_\ell'(\cos\theta) = \frac{d}{d\theta}P_\ell(\cos\theta)$

$\Rightarrow$ (1) $A_1\cdot a = -H_0 a + \frac{B_1}{a^2}$

(2) $\mu A_1 = -\mu_0 H_0 - \frac{2B_1\mu_0}{a^3}$

$$-H_0 + \frac{B_1}{a^3} = -\frac{\mu_0}{\mu} H_0 - \frac{2B_1}{a^3}\frac{\mu_0}{\mu}$$

$$\Rightarrow \frac{B_1}{a^3}\left(1 + 2\frac{\mu_0}{\mu}\right) = H_0\left(1 - \frac{\mu_0}{\mu}\right) \Rightarrow B_1 = H_0 a^3 \frac{\mu - \mu_0}{\mu + 2\mu_0}$$

$$A_1 = -H_0 + \frac{B_1}{a^3} \Rightarrow A_1 = -\frac{3\mu_0}{\mu + 2\mu_0} H_0 .$$

$$\Rightarrow \Phi_{\text{inside}} = -\frac{3\mu_0}{\mu + 2\mu_0} H_0 r\cos\theta = -\frac{3\mu_0}{\mu + 2\mu_0} H_0 z$$

$$\Phi_{\text{outside}} = -H r\cos\theta + H_0 a^3 \frac{\mu - \mu_0}{\mu + 2\mu_0}\frac{1}{r^2}\cos\theta$$

& one can find

$$\vec{H}_{\text{inside}} = -\nabla\Phi_{\text{inside}} = \frac{3\mu_0}{\mu + 2\mu_0}\vec{H}_0$$

$$\vec{B}_{\text{inside}} = \mu\vec{H}_{\text{inside}}$$

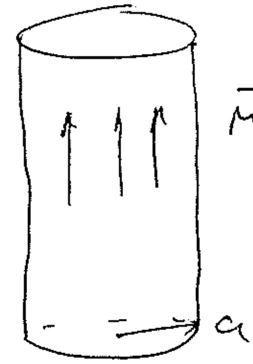
Effective magnetization: $\vec{M} = \frac{1}{\mu_0}\vec{B}_{\text{inside}} - \vec{H}_{\text{inside}}$

$$= \left(\frac{\mu}{\mu_0} - 1\right)\vec{H}_{\text{inside}} \Rightarrow \vec{M} = 3\frac{\mu - \mu_0}{\mu + 2\mu_0}\vec{H}_0$$

cf. Jackson (5.115).

<u>Example</u>: infinite cylindrical bar magnet:

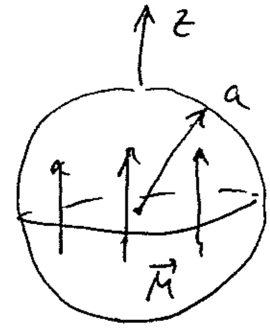
find $\vec{B}$, $\vec{H}$.



$$\Phi_M = \frac{1}{4\pi}\oint_S da' \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} = 0$$

$$\Rightarrow \vec{H} = 0, \quad \vec{B}_{\text{inside}} = \mu_0(\vec{H} + \vec{M}) = \mu_0\vec{M}, \quad \vec{B}_{\text{outside}} = 0.$$

Example: Ferromagnetic sphere, take 2.



$\vec{M} = M\hat{z}$ inside the sphere

$\Rightarrow \vec{M} = M\hat{z}\,\theta(a-r)$.

use
$$\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \;\frac{\vec{\nabla}'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$\vec{\nabla}\times\vec{M} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & M_z \end{vmatrix} = \hat{x}\,\partial_y M_z - \hat{y}\,\partial_x M_z$$

$$= \hat{x}\,M\,\partial_y\,\theta\left(a-\sqrt{x^2+y^2+z^2}\right) - \hat{y}\,M\,\partial_x\,\theta\left(a-\sqrt{x^2+y^2+z^2}\right) =$$

$$= \hat{x}\,M\,\delta(a-r)\,\frac{-y}{r} - \hat{y}\,M\,\delta(a-r)\,\frac{-x}{r} =$$

$$= \left[-\hat{x}\sin\theta\sin\varphi + \hat{y}\sin\theta\cos\varphi\right]M\,\delta(a-r) = M\sin\theta\,\delta(r-a)\cdot \underbrace{\left[-\hat{x}\sin\varphi + \hat{y}\cos\varphi\right]}_{\hat{\varphi}}$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int_0^\infty dr'\,r'^2\int_{-1}^{1} d\cos\theta'\int_0^{2\pi} d\varphi' \cdot$$

$$\cdot \underbrace{4\pi\sum_{l,m}\frac{1}{2l+1}\,\frac{r_<^l}{r_>^{l+1}}\,Y_{lm}^*(\theta',\varphi')\,Y_{lm}(\theta,\varphi)}_{\frac{1}{|\vec{x}-\vec{x}'|}} \cdot M\,\delta(r'-a)\sin\theta'\,\hat{\varphi}'$$

now $r_{\gtrless} = {\max \atop \min}\{r, a\}$

$$= M\mu_0 a^2\sum_{l,m}\frac{1}{2l+1}\,\frac{r_<^l}{r_>^{l+1}}\int_{-1}^{1} d\cos\theta'\int_0^{2\pi} d\varphi'\,\sin\theta'\,\hat{\varphi}'\,Y_{lm}^*(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$

$$= \mu_0 M a^2 \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\varphi' \, Y_{lm}^*(\theta',\varphi') \cdot$$

$$\underbrace{\begin{pmatrix} -\sin\theta' \sin\varphi' \\ \sin\theta' \cos\varphi' \\ 0 \end{pmatrix}}_{\parallel} \Rightarrow$$

$$\begin{pmatrix} +\sqrt{\frac{8\pi}{3}} \left(Y_{11} + Y_{1,-1} \right) \cdot \frac{1}{2i} \\ -\sqrt{\frac{8\pi}{3}} \left(Y_{11} - Y_{1,-1} \right) \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{A} = \mu_0 M a^2 \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) \begin{pmatrix} \frac{1}{2i}\sqrt{\frac{8\pi}{3}} \delta_{l1} (\delta_{m1} + \delta_{m,-1}) \\ -\frac{1}{2}\sqrt{\frac{8\pi}{3}} \delta_{l1} (\delta_{m1} - \delta_{m,-1}) \\ 0 \end{pmatrix}$$

$$= \mu_0 M a^2 \frac{1}{3} \frac{r_<}{r_>^2} \begin{pmatrix} -\sin\theta \sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix} = \sin\theta\,\hat{\varphi}$$

$$\Rightarrow \boxed{\vec{A} = \frac{\mu_0 M a^2}{3} \frac{r_<}{r_>^2} \sin\theta\,\hat{\varphi}}$$

$\Rightarrow \vec{A}_{out} = \frac{\mu_0 M a^3}{3} \frac{\sin\theta}{r^2} \hat{\varphi}$ ~ dipole

$\vec{A}_{in} = \frac{\mu_0 M}{3} r\sin\theta\,\hat{\varphi} = \frac{\mu_0 M}{3} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$

$\Rightarrow \vec{B}_{in} = \vec{\nabla} \times \vec{A}_{in} = \frac{2}{3}\mu_0 M \hat{z}$ ~ unif. field

$\Rightarrow$ can find $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ ~ same as before