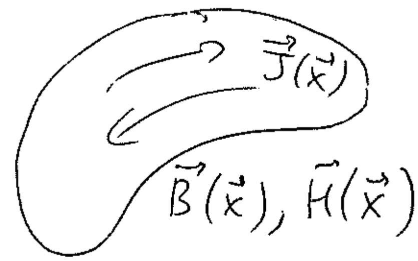
$$\oint_C \vec{E}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{E})\cdot\hat{n}\,da = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\,da$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$ differential form of Faraday's law (generalizes $\vec{\nabla}\times\vec{E}=0$ of electrostatics).

Energy in the Magnetic Field.

Energy change rate is (for a point charge)



$$\frac{dW}{dt} = \vec{V}\cdot\vec{F} = q\,\vec{V}\cdot\vec{E}$$

(point charge q moving with velocity $\vec{v}$) $\quad \vec{\nabla}\cdot\vec{J}=0 \Rightarrow$ can break the current into small loops.

$\Rightarrow \delta W = -\delta t \cdot \vec{J}\cdot\vec{E}$ ← as $\vec{J}=q\vec{v}$ ← work done to bring in a small current loop a

Let's change $\vec{B}$ by $\delta\vec{B}(\vec{x})$: $\left(\vec{\nabla}\times\vec{E} = -\frac{\delta\vec{B}}{\delta t}\right)$

$$\delta W = -\delta t\int d^3x\;\vec{J}\cdot\vec{E} = -\delta t\int d^3x\,(\vec{\nabla}\times\vec{H})\cdot\vec{E}$$

As $\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H}) \Rightarrow$

$$\delta W = +\delta t\int d^3x\left\{\underbrace{\vec{\nabla}(\vec{E}\times\vec{H})}_{\text{surface integral}} - \vec{H}\cdot\vec{\nabla}\times\vec{E}\right\}$$

$$= -\delta t\int d^3x\;\vec{H}\cdot(\vec{\nabla}\times\vec{E}) \Rightarrow \text{as } \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

(note also that $\delta\vec{B} = \frac{\partial\vec{B}}{\partial t}\delta t + (\underbrace{\delta\vec{x}}_{\text{coordinates don't change}; \,=0}\cdot\vec{\nabla})\vec{B} = \frac{\partial\vec{B}}{\partial t}\delta t$.)

$$\Rightarrow \boxed{\delta W = \int d^3x \; \vec{H} \cdot \delta \vec{B}}$$

If $\vec{B} = \mu \vec{H} \Rightarrow \boxed{W = \frac{1}{2} \int d^3x \; \vec{H} \cdot \vec{B}}$

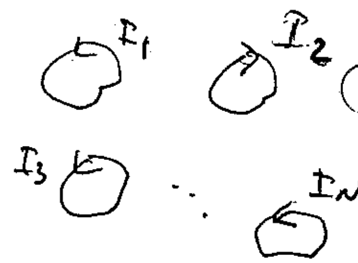
Alternatively, as $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow$

$$W = \frac{1}{2} \int d^3x \; \vec{H} \cdot (\vec{\nabla} \times \vec{A}) = \frac{1}{2} \int d^3x \; \vec{A} \cdot \underbrace{(\vec{\nabla} \times \vec{H})}_{= \vec{J}}$$

$$\Rightarrow \boxed{W = \frac{1}{2} \int d^3x \; \vec{A} \cdot \vec{J}}$$

Self- and Mutual Inductances.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \; \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \Rightarrow$$



$$\Rightarrow W = \frac{\mu_0}{8\pi} \int d^3x \, d^3x' \; \frac{\vec{J}(\vec{x}) \cdot \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

If we have N circuits with currents $I_1, \ldots, I_N$:

$$W = \frac{1}{2} \sum_{i=1}^{N} L_i I_i^2 + \frac{1}{2} \sum_{i \neq j} M_{ij} I_i I_j$$

Definition $L_i \sim$ self-inductance

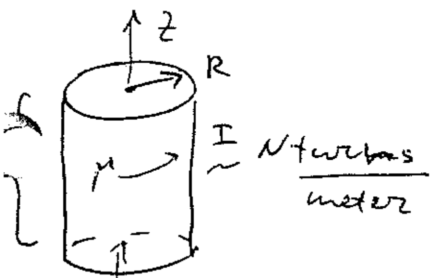
$M_{ij} \sim$ mutual inductance between i & j

as $W = \frac{\mu_0}{8\pi} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{V_i} d^3x_i \int_{V_j} d^3x_j' \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_j')}{|\vec{x}_i - \vec{x}_j'|}$

$$\Rightarrow \quad L_i = \frac{\mu_0}{4\pi I_i^2} \int_{V_i} d^3x_i \int_{V_i} d^3x_i' \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_i')}{|\vec{x}_i - \vec{x}_i'|}$$

$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_{V_i} d^3x_i \int_{V_j} d^3x_j' \frac{\vec{J}(\vec{x}_i) \cdot \vec{J}(\vec{x}_j')}{|\vec{x}_i - \vec{x}_j'|}$$

<u>Example</u>: self-inductance of a solenoid:



filled with material with permeability μ.

$\vec{\nabla} \times \vec{H} = \vec{J} \Rightarrow \vec{H}_{in} = N I \hat{z}, \ \vec{H}_{out} = 0$

~ $\vec{B}_{in} = \mu \vec{H}_{in} = \mu N I \hat{z}$

$\Rightarrow W = \frac{1}{2} \int d^3x \ \vec{B} \cdot \vec{H} = \frac{1}{2} \mu N^2 I^2 \cdot V$

$V = \pi R^2 \cdot h$ ~ volume

$$\Rightarrow W = \frac{1}{2} L I^2 = \frac{1}{2} \mu N^2 I^2 \pi R^2 h$$

$$\Rightarrow \boxed{L = \mu N^2 \pi R^2 h} = \mu N^2 V$$

Linear currents: $\vec{J} d^3x = I d\vec{l} \Rightarrow W = \frac{1}{2} \int d^3x \ \vec{J} \cdot \vec{A} = \frac{1}{2} \oint \vec{A} \cdot d\vec{l} \cdot I =$

$= I \cdot \frac{1}{2} \int da \cdot \hat{n} \underbrace{(\vec{\nabla} \times \vec{A})}_{= \vec{B}} = \frac{1}{2} I \underset{\text{magnetic flux}}{\Phi} = \frac{1}{2} L I^2 \Rightarrow \boxed{L = \frac{\Phi}{I}}$

$\Rightarrow \mathcal{E} \overset{\text{Lenz's Law}}{=} - \frac{d\Phi}{dt} = - L \frac{dI}{dt} = - L \dot{I}$ ~ also familiar

formula known from undergraduate E&M.

For simple linear loops:

$$\vec{J}\, d^3x \rightarrow I\, d\vec{\ell}$$

$$\Rightarrow \boxed{\begin{aligned} L_i &= \frac{\mu_0}{4\pi} \oint_{C_i} d\vec{\ell}_i \cdot \oint_{C_i} d\vec{\ell}_i' \, \frac{1}{|\vec{x}_i - \vec{x}_i'|} \\ M_{ij} &= \frac{\mu_0}{4\pi} \oint_{C_i} d\vec{\ell}_i \cdot \oint_{C_j} d\vec{\ell}_j \, \frac{1}{|\vec{x}_i - \vec{x}_j|} \end{aligned}}$$

$$\boxed{W = \frac{\mu_0}{8\pi} \sum_{i,j=1}^{N} \left[\oint_{C_i} \oint_{C_j} \frac{d\vec{\ell}_i \cdot d\vec{\ell}_j}{|\vec{x}_i - \vec{x}_j|} \right] I_i I_j}$$

Lenz's Law: start with Faraday's law,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \text{electromotive force}$$

$$\mathcal{E} = \oint_C d\vec{\ell} \cdot \vec{E} = \int_S da\, \hat{n} \cdot (\vec{\nabla} \times \vec{E}) = -\int_S da\, \hat{n} \cdot \frac{\partial \vec{B}}{\partial t} =$$

$$= -\frac{d}{dt} \underbrace{\int_S da\, \hat{n} \cdot \vec{B}}_{\text{flux } \Phi \text{ of } \vec{B}} = -\frac{d\Phi}{dt} \Rightarrow \boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

Lenz's law