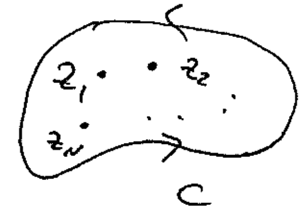
Last class | Complex Analysis 101 (cont'd)

Proved the residue theorem:



If $f(z)$ is analytic except for a finite # of singularities $z_1, \ldots, z_N$, then

$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{N} \text{Res}\, f(z_i)}$$

$\text{Res}\, f(z_i)$ is the coefficient a_{-1} of the Laurent expansion of $f(z)$ around z_i.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$ ~ Taylor series (analytic $f(z)$ at z_0)

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$ ~ Laurent series. $f(z)$ may not be analytic at z_0.

(if all $a_{-n} \neq 0 \Rightarrow$ essential singularity)

Simple pole: $f(z) = \frac{a_{-1}}{z-z_0} + a_0 + \ldots$

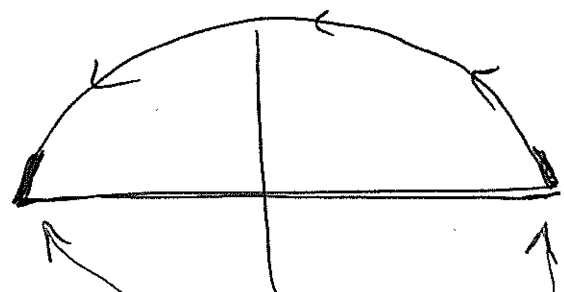
$$\Rightarrow \boxed{\text{Res}\, f(z_0) = \lim_{z \to z_0} \left[(z-z_0) f(z) \right]}$$

Pole of order n: $f(z) = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{z-z_0} + a_0 + \dots$

$$\boxed{\mathrm{Res}\, f(z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right]}$$

for $n=1$ reduces to the simple pole expressio

Integrals with complex exponentials:



$\int_{-\infty}^{\infty} dx\, f(x)\, e^{ik \cdot x}$, $k > 0$

can close the contour

in the upper half-plane

if $\lim_{R \to \infty} f(Re^{i\varphi}) = 0$ for all φ.

(Jordan's lemma).

$e^{ikx} = e^{ikRe^{i\varphi}} = e^{ikR\cos\varphi - kR\sin\varphi}$

$\Rightarrow$ if $\sin\varphi \lesssim \frac{1}{kR} = \sin\varphi_0$ the exponential won't

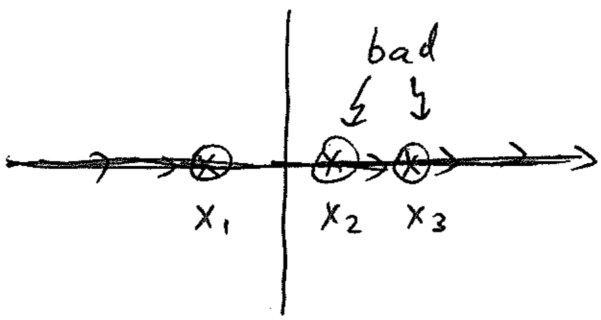
fall off (see segments on the plot)

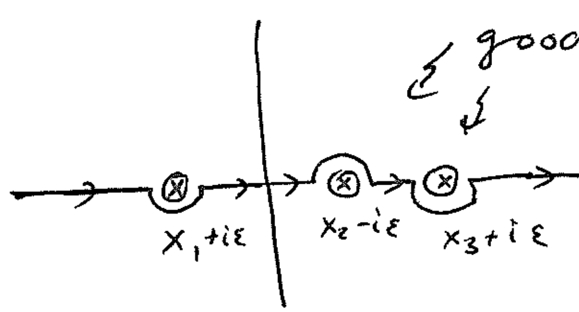
$\Rightarrow \lim_{R\to\infty} \int_{C_R} dz\, f(z) e^{ikz} \simeq \lim_{R\to\infty} 2Ri \underbrace{\int^{\varphi_0} d\varphi}_{\sim\varphi_0 \sim \frac{1}{R}} e^{i\varphi} f(Re^{i\varphi})\, e^{ikR}$

$\Rightarrow$ need $f(Re^{i\varphi}) \to 0$ as $R \to 0$ for the integral $= 0$.

Integration along real axis:

When integrating over the real axis, may run into situation when the poles are on the real axis. The integral is then ill-defined. One can specify how to go around those poles, thus defining the integral.





$$\int_{-\infty}^{\infty} dx \, \frac{1}{(x-x_1)(x-x_2)(x-x_3)} \Rightarrow \text{the integral has no value}$$

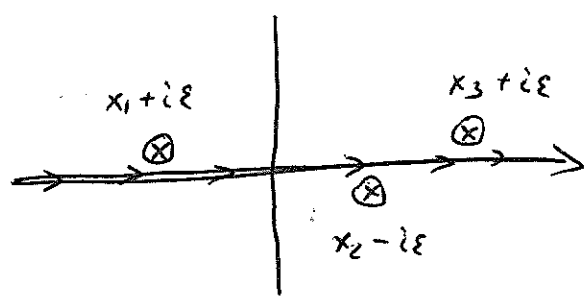
↑ ill-defined!

To specify how to go around the pole we replace

$$\frac{1}{x - x_0} \rightarrow \frac{1}{x - x_0 \pm i\varepsilon}$$

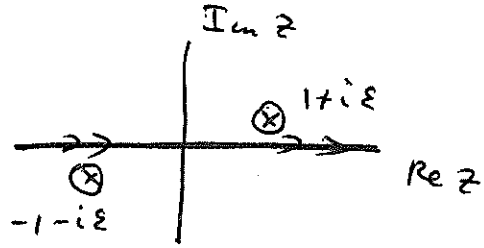
where ε is infinitesimal, and $\pm$ defines whether the contour goes below/above the pole.

One can draw this as moving the poles, not the contour (by infinitesimal quantities)



Example $\displaystyle\int_{-\infty}^{\infty} dx \, \frac{1}{(x-1)(x+1)}$ is ill-defined.

However, for instance,



$$\int_{-\infty}^{\infty} dx \, \frac{1}{(x-1-i\varepsilon)(x+1+i\varepsilon)}$$

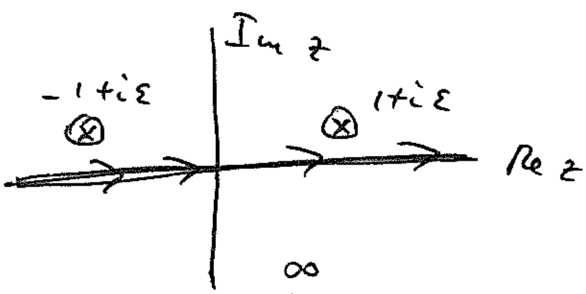
is well-defined. Closing the contour in upper half-plane we get

$$\int_{-\infty}^{\infty} dx \, \frac{1}{(x-1-i\varepsilon)(x+1+i\varepsilon)} = 2\pi i \, \frac{1}{2+2i\varepsilon} \xrightarrow[\varepsilon \to 0]{} \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} dx \, \frac{1}{(x-1-i\varepsilon)(x+1+i\varepsilon)} = \pi i .$$

We can go around the poles in different ways, each time obtaining <u>different</u> (!) integrals!

Example $\displaystyle\int_{-\infty}^{\infty} dx \, \frac{1}{(x-1-i\varepsilon)(x+1-i\varepsilon)} =$ | close contour in the lower half-plane



$$= -2\pi i \cdot \underset{\uparrow}{\emptyset} = 0$$

no poles in the lower half-plane

$$\Rightarrow \int_{-\infty}^{\infty} dx \, \frac{1}{(x-1-i\varepsilon)(x+1-i\varepsilon)} = 0. \quad \text{(cf. previous example)}$$

Def. Principle value regularization is defined by:

$$\boxed{PV\,\frac{1}{x} \equiv \frac{1}{2}\left[\frac{1}{x-i\varepsilon} + \frac{1}{x+i\varepsilon}\right]}\,, \qquad x = \text{real}.$$

$PV\,\frac{1}{x} = \frac{1}{2}\Big[$  $+$  $\Big]$ ~ half-sum of the above & below contours.

Note that $PV\,\frac{1}{x} = \frac{1}{2}\left[\frac{1}{x-i\varepsilon} + \frac{1}{x+i\varepsilon}\right] = \frac{x}{x^2+\varepsilon^2}$ ~ real!

Sometimes people use $P\frac{1}{x}$ to denote $PV\frac{1}{x}$.

Consider $\quad \frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = \frac{2i\varepsilon}{x^2+\varepsilon^2} \xrightarrow[\varepsilon\to 0]{} \begin{cases} 0, & x\neq 0 \\ \infty, & x=0 \end{cases}$

($x \in$ Reals)

$\Rightarrow$ could be a δ-function. To check integrate:

$$\int_{-\infty}^{\infty} dx\left[\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}\right] = 2i\varepsilon\int_{-\infty}^{\infty}\frac{dx}{(x-i\varepsilon)(x+i\varepsilon)} = \cancel{2i\varepsilon}\,2\pi i\,\frac{1}{\cancel{2i\varepsilon}} = 2\pi i$$

$$\Rightarrow \quad \lim_{\varepsilon\to 0}\left[\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}\right] = 2\pi i\,\delta(x).$$

Usually people drop the limit sign and write

$$\boxed{\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = 2\pi i\,\delta(x).}$$

Note that

$$\boxed{\begin{aligned} \frac{1}{x-i\varepsilon} &= PV\frac{1}{x} + \pi i\,\delta(x) \\ \frac{1}{x+i\varepsilon} &= PV\frac{1}{x} - \pi i\,\delta(x) \end{aligned}}$$

This follows from the definition of PV and the formula for δ-function we've obtained.

Example $\int_{-\infty}^{\infty} dx\; PV\frac{1}{x}\; PV\frac{1}{x-1} = \int_{-\infty}^{\infty} dx\; \frac{1}{2}\left[\frac{1}{x-i\varepsilon} + \frac{1}{x+i\varepsilon}\right]$

$$\cdot \frac{1}{2}\left[\frac{1}{x-1-i\varepsilon} + \frac{1}{x-1+i\varepsilon}\right] = \frac{1}{4}\int_{-\infty}^{\infty} dx \left[\frac{1}{x-i\varepsilon}\,\frac{1}{x-1+i\varepsilon} + \frac{1}{x+i\varepsilon}\,\frac{1}{x-1-i\varepsilon}\right]$$

$$= \frac{1}{4}\left[2\pi i\,(-1) + 2\pi i\right] = 0.$$

Example (higher-order poles)

$$\int_{-\infty}^{\infty} dx\,\frac{e^{ix}}{(x^2+1)^2} = \int_{-\infty}^{\infty} dx\,\frac{e^{ix}}{(x-i)^2(x+i)^2} =$$

close the contour in the upper half-plane

$$= 2\pi i\,\frac{1}{(2-1)!}\,\lim_{x\to i}\frac{d}{dx}\left[(x-i)^2\frac{e^{ix}}{(x-i)^2(x+i)^2}\right] = 2\pi i\,\cdot$$

$$\cdot \lim_{x\to i}\frac{d}{dx}\left(\frac{e^{ix}}{(x+i)^2}\right) = 2\pi i\,\lim_{x\to i}\left[\frac{i\,e^{ix}}{(x+i)^2} - 2\,\frac{e^{ix}}{(x+i)^3}\right] = 2\pi i\cdot e^{-1}\cdot$$

$$\cdot\left[\frac{i}{-4} - 2\,\frac{i}{8}\right] = 2\pi i\,e^{-1}\,\frac{-i}{2} = \frac{\pi}{e}$$