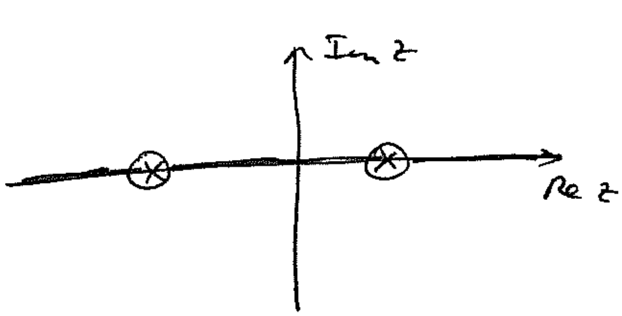
Last time | **Integrals along real axis**

What do we do if we get an integral along the real axis, which has poles along the real axis?



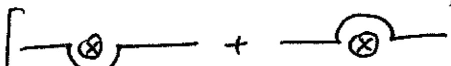
$\Rightarrow$ Strictly-speaking it's ill-defined.

A well defined integral can be (but does not have to be) constructed by distorting the contour above or below the poles:

$$\frac{1}{x-x_0} \to \frac{1}{x-x_0-i\varepsilon}$$

$$\frac{1}{x-x_0} \to \frac{1}{x-x_0+i\varepsilon}$$

Def. Principal value: $\boxed{PV\,\frac{1}{x} \equiv \frac{1}{2}\left[\frac{1}{x-i\varepsilon} + \frac{1}{x+i\varepsilon}\right]}$

$= \frac{1}{2}\Big[$  $\Big]$

We showed that

$$\boxed{\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = 2\pi i\,\delta(x).}$$

This way:
$$\boxed{\begin{aligned} \frac{1}{x - i\varepsilon} &= PV\,\frac{1}{x} + \pi i\,\delta(x) \\ \frac{1}{x + i\varepsilon} &= PV\,\frac{1}{x} - \pi i\,\delta(x) \end{aligned}}$$

Def. Principle value regularization is defined by:

$$\boxed{PV\,\frac{1}{X} \equiv \frac{1}{2}\left[\frac{1}{X-i\varepsilon} + \frac{1}{X+i\varepsilon}\right]}\,, \qquad X = \text{real}.$$

$PV\,\frac{1}{x} = \frac{1}{2}\Big[$  $+$  $\Big]$ ~ half-sum of the above & below contours.

Note that $PV\,\frac{1}{x} = \frac{1}{2}\left[\frac{1}{x-i\varepsilon} + \frac{1}{x+i\varepsilon}\right] = \frac{x}{x^2+\varepsilon^2}$ ~ real!

Sometimes people use $P\,\frac{1}{x}$ to denote $PV\,\frac{1}{x}$.

Consider $\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = \frac{2i\varepsilon}{x^2+\varepsilon^2} \xrightarrow[\varepsilon\to 0]{} \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$

$(x \in \text{Reals})$

$\Rightarrow$ could be a δ-function. To check integrate:

$$\int_{-\infty}^{\infty} dx \left[\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}\right] = 2i\varepsilon \int_{-\infty}^{\infty} \frac{dx}{(x-i\varepsilon)(x+i\varepsilon)} = \cancel{2i\varepsilon}\, 2\pi i \frac{1}{\cancel{2i\varepsilon}} = 2\pi i$$

$$\Rightarrow \lim_{\varepsilon\to 0}\left[\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}\right] = 2\pi i\,\delta(x).$$

Usually people drop the limit sign and write

$$\boxed{\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = 2\pi i\,\delta(x).}$$

Note that

$$\boxed{\begin{aligned} \frac{1}{x-i\varepsilon} &= PV\,\frac{1}{x} + \pi i\,\delta(x) \\ \frac{1}{x+i\varepsilon} &= PV\,\frac{1}{x} - \pi i\,\delta(x) \end{aligned}}$$

This follows from the definition of PV and the formula for δ- function we've obtained.

<u>Example</u> $\displaystyle\int_{-\infty}^{\infty} dx\; PV\frac{1}{x}\; PV\frac{1}{x-1} = \int_{-\infty}^{\infty} dx\; \frac{1}{2}\left[\frac{1}{x-i\varepsilon} + \frac{1}{x+i\varepsilon}\right]$

$$\cdot\frac{1}{2}\left[\frac{1}{x-1-i\varepsilon} + \frac{1}{x-1+i\varepsilon}\right] = \frac{1}{4}\int_{-\infty}^{\infty} dx \left[\frac{1}{x-i\varepsilon}\,\frac{1}{x-1+i\varepsilon} + \frac{1}{x+i\varepsilon}\,\frac{1}{x-1-i\varepsilon}\right]$$

$$= \frac{1}{4}\left[2\pi i\,(-1) + 2\pi i\right] = 0.$$

<u>Example (higher-order poles)</u>

$$\int_{-\infty}^{\infty} dx\; \frac{e^{ix}}{(x^2+1)^2} = \int_{-\infty}^{\infty} dx\; \frac{e^{ix}}{(x-i)^2(x+i)^2} =$$

close the contour in the upper half-plane

$$= 2\pi i\,\frac{1}{(2-1)!}\,\lim_{x\to i}\frac{d}{dx}\left[(x-i)^2\frac{e^{ix}}{(x-i)^2(x+i)^2}\right] = 2\pi i\,\cdot$$

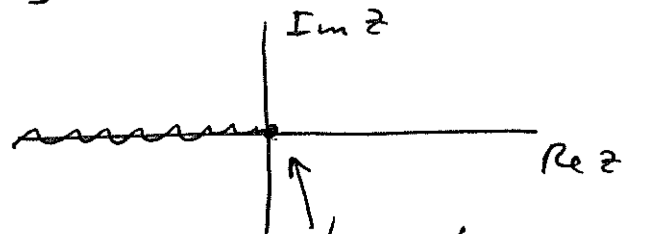
$$\cdot\lim_{x\to i}\frac{d}{dx}\left(\frac{e^{ix}}{(x+i)^2}\right) = 2\pi i\,\lim_{x\to i}\left[\frac{i e^{ix}}{(x+i)^2} - 2\,\frac{e^{ix}}{(x+i)^3}\right] = 2\pi i\cdot e^{-1}\cdot$$

$$\cdot\left[\frac{i}{-4} - 2\,\frac{i}{8}\right] = 2\pi i\, e^{-1}\,\frac{-i}{2} = \frac{\pi}{e}$$

Branch cuts: poles are not the only singularities, one may have **branch cuts** as well: these are the curves in the complex plane across which the (otherwise) analytic function is discontinuous.

Example $f(z) = \sqrt{z} \Rightarrow \sqrt{1} = \pm 1 \sim$ multi-valued.

Note that $\sqrt{-1} = \pm i \sim$ also ambiguous. One has to (often by convention) specify where the function "jumps". For $\sqrt{z}$ the conventional branch cut is for $z \in [-\infty, 0]$:



Hence, $\sqrt{i} = e^{i\frac{\pi}{4}}$,

$$\sqrt{-i} = e^{-i\frac{\pi}{4}},$$

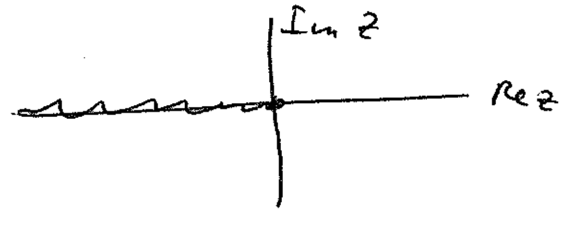
$$\sqrt{-1+i\varepsilon} = i, \quad \sqrt{-1-i\varepsilon} = -i$$

ε ↑ infinitesimal

Example $f(z) = \ln z \equiv \ln|z| + i \operatorname{Arg} z$

That is, if $z = \rho e^{i\varphi} \Rightarrow \ln z = \ln\rho + i\varphi$ as expected.

$\ln z$ has the same branch cut as $\sqrt{z}$: $z \in (-\infty, 0]$.



(also chosen by convention)

$$\ln i = i\frac{\pi}{2}, \quad \ln(-1+i\varepsilon) = i\pi, \quad \ln(-1-i\varepsilon) = -i\pi$$

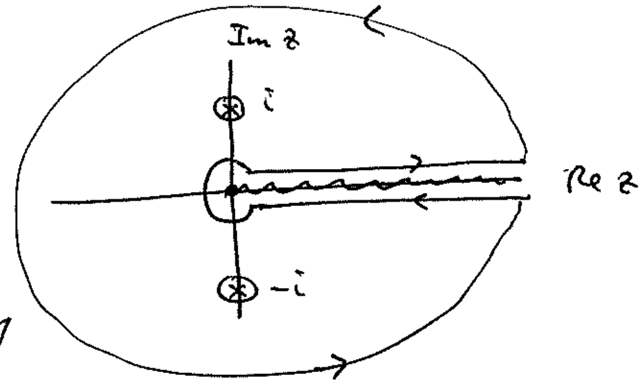
Example $z^p = e^{p \ln z}$ ⇒ same branch cut as $\ln z$

$$I = \int_0^\infty dx \, \frac{x^p}{x^2+1} \,, \quad 0 < p < 1$$

assume branch cut along $\mathbb{R}^+$:

$0 \leq \varphi \leq 2\pi$, $z = \rho e^{i\varphi}$

Consider the contour C



$$\oint_C dz \, \frac{z^p}{z^2+1} = \int_0^\infty dx \, \frac{x^p}{x^2+1} + \int_\infty^0 dx \, \frac{(x \cdot e^{2\pi i})^p}{x^2+1} = \left(1 - e^{2p i\pi}\right) I$$

$$\Rightarrow I = \frac{1}{1 - e^{2\pi i p}} \oint_C dz \, \frac{z^p}{\underbrace{z^2+1}_{(z-i)(z+i)}} =$$

inside C we have poles at $z = i = e^{i\pi/2}$ and $z = -i = e^{i\frac{3}{2}\pi}$

$$\Rightarrow I = \frac{1}{1 - e^{2\pi i p}} \cdot 2\pi i \left[\frac{1}{2i} e^{\frac{i\pi p}{2}} - \frac{1}{2i} e^{i\frac{3}{2}\pi p} \right] =$$

$$= \pi \, \frac{e^{-\frac{i\pi p}{2}} - e^{\frac{i\pi p}{2}}}{e^{-i\pi p} - e^{i\pi p}} = \pi \, \frac{-2i \sin\left(\frac{\pi p}{2}\right)}{-2i \sin(\pi p)} = \pi \, \frac{\sin\left(\frac{\pi p}{2}\right)}{\sin(\pi p)}$$

$$\Rightarrow \boxed{I = \frac{\pi}{2 \cos\left(p\pi/2\right)}}$$

⇒ we've employed the branch cut for our advantage

Other useful reference: M. J. Ablowitz, A. S. Fokas "Complex Variables"