Plane Electromagnetic Waves

(in infinite LIH media)

no sources ⇒

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t} = 0$$

$$\vec{D} = \varepsilon\vec{E}\,,\ \vec{B} = \mu\vec{H} \Rightarrow \vec{\nabla}\times\vec{B} - \mu\varepsilon\frac{\partial\vec{E}}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot(\underbrace{\vec{\nabla}\cdot\vec{B}}_{0}) - \nabla^2\vec{B} = -\nabla^2\vec{B} = \mu\varepsilon\frac{\partial\vec{\nabla}\times\vec{E}}{\partial t} = -\mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0}$$

Similarly, $\boxed{\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0}$

In general, write $\vec{E}(\vec{x},t) = \int\frac{d^3k\,d\omega}{(2\pi)^4}\vec{E}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

and $\vec{B}(\vec{x},t) = \int\frac{d^3k\,d\omega}{(2\pi)^4}\vec{B}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$$\Rightarrow -\vec{k}^2 + \mu\varepsilon\omega^2 = 0 \Rightarrow \omega = \pm\frac{1}{\sqrt{\mu\varepsilon}}|\vec{k}|$$

Phase velocity of a wave $v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$ Speed of wave crest

with the index of refraction $n = \sqrt{\frac{\mu\,\varepsilon}{\mu_0\,\varepsilon_0}}$

$\underbrace{\vec{k}\cdot\vec{x} - \omega t}_{\text{phase}} = \text{const} \Rightarrow$ get $v = \frac{\omega}{k}\left(= \frac{d\vec{x}}{dt}\right)$.

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

General solution (for real $\vec{E}$ & $\vec{B}$):

$\vec{E}(\vec{k},\omega)$ on prev. page.

$$\boxed{\begin{aligned}\vec{E}(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\left[\vec{E}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \vec{E}^*(\vec{k},\omega)\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right] \\ \vec{B}(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\left[\vec{B}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \vec{B}^*(\vec{k},\omega)\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right]\end{aligned}}$$

$\omega_k = k/\sqrt{\mu\varepsilon}$

complex conjugate of 1st term

cf. in 1-dim solution of wave equation

is $u(x,t) = f(x-vt) + g(x+vt)$

(can always redefine $\vec{k}\to-\vec{k}$ in the 2nd term $\Rightarrow$ get $\omega t+\vec{k}\cdot\vec{x}$ argument)

<u>Monochromatic plane wave:</u> (fix one mode $\vec{k}$)

$$\begin{cases}\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{x}-\omega t) \\ \vec{B} = \vec{B}_0 \cos(\vec{k}\cdot\vec{x}-\omega t)\end{cases}$$

$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0$

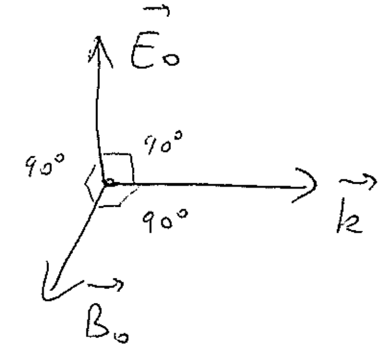
$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B}_0 = 0$

$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \vec{k}\times\vec{E}_0 - \omega\vec{B}_0 = 0$

$\Rightarrow \vec{B}_0 = \frac{1}{\omega}\vec{k}\times\vec{E}_0$

$\vec{\nabla}\times\vec{B} - \mu\varepsilon\frac{\partial\vec{E}}{\partial t} = 0 \Rightarrow \vec{k}\times\vec{B}_0 + \mu\varepsilon\,\omega\vec{E}_0 = 0$

$\Rightarrow \vec{E}_0 = -\frac{\omega}{\vec{k}^2}\vec{k}\times\vec{B}_0$



$\Rightarrow \vec{k}, \vec{E}_0$ & $\vec{B}_0$ are orthogonal to each other! $|\vec{B}_0| = \sqrt{\mu\varepsilon}\,|\vec{E}_0|$