**Last time** Solution of Maxwell Equations in Lorenz Gauge

SI units:

$$\boxed{\begin{aligned} \Phi(\vec{x}, t) &= \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \rho\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right) \\ \vec{A}(\vec{x}, t) &= \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \vec{J}\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right). \end{aligned}}$$

## Plane Electromagnetic Waves (cont'd)

LIH medium:

$$\boxed{\left(\vec{\nabla}^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0}$$

(no sources, $\rho = 0$, $\vec{J} = 0$)

Solution:

$$\boxed{\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix}(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} e^{-i\omega t + i\vec{k}\cdot\vec{x}} \begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix}(\vec{k}) + c.c.}$$

$$\omega = \frac{k}{\sqrt{\mu\varepsilon}}$$

Phase velocity $v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \Rightarrow \quad v = \frac{c}{n} \quad \text{with} \quad n = \sqrt{\frac{\mu \varepsilon}{\mu_0 \varepsilon_0}}$$

index of refraction

# Plane Electromagnetic Waves

(in infinite LIH media)

no sources $\Rightarrow$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t} = 0$$

$$\vec{D} = \varepsilon\vec{E}, \quad \vec{B} = \mu\vec{H} \Rightarrow \vec{\nabla}\times\vec{B} - \mu\varepsilon\frac{\partial \vec{E}}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{0} - \nabla^2\vec{B} = -\nabla^2\vec{B} = \mu\varepsilon\frac{\partial\vec{\nabla}\times\vec{E}}{\partial t} = -\mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0}$$

Similarly, $\boxed{\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0}$

In general, write $\vec{E}(\vec{x},t) = \int\frac{d^3k\,d\omega}{(2\pi)^4}\vec{E}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

and $\vec{B}(\vec{x},t) = \int\frac{d^3k\,d\omega}{(2\pi)^4}\vec{B}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$$\Rightarrow -\vec{k}^2 + \mu\varepsilon\omega^2 = 0 \Rightarrow \omega = \pm\frac{1}{\sqrt{\mu\varepsilon}}|\vec{k}|$$

Phase velocity of a wave $V = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$ Speed of wave crest

with the index of refraction $n = \sqrt{\frac{\mu\,\varepsilon}{\mu_0\varepsilon_0}}$

$\underbrace{\vec{k}\cdot\vec{x} - \omega t}_{\text{phase}} = \text{const} \Rightarrow$ get $V = \frac{\omega}{k}\left(= \frac{d\vec{x}}{dt}\right)$.

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

General solution (for real $\vec{E}$ & $\vec{B}$): $\vec{E}(\vec{k},\omega)$ on prev. page.

$$\boxed{\begin{aligned}\vec{E}(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\left[\vec{E}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \vec{E}^*(\vec{k},\omega)\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right] \\ \vec{B}(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\left[\vec{B}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \underbrace{\vec{B}^*(\vec{k},\omega)\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}}_{\text{complex conjugate of 1st term}}\right]\end{aligned}}$$

$\omega_k = k/\sqrt{\mu\varepsilon}$

cf. in 1-dim solution of wave equation

is $u(x,t) = f(x-vt) + g(x+vt)$

(can always redefine $\vec{k} \to -\vec{k}$ in the 2nd term $\Rightarrow$ get $\omega t + \vec{k}\cdot\vec{x}$ argument)

<u>Monochromatic plane wave:</u> (fix one mode $\vec{k}$)

$$\begin{cases}\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{x}-\omega t) \\ \vec{B} = \vec{B}_0 \cos(\vec{k}\cdot\vec{x}-\omega t)\end{cases}$$

$$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B}_0 = 0$$

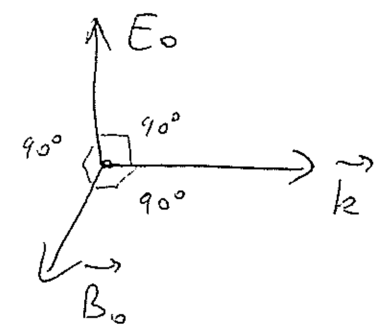
$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \vec{k}\times\vec{E}_0 - \omega\vec{B}_0 = 0$$

$$\Rightarrow \vec{B}_0 = \frac{1}{\omega}\vec{k}\times\vec{E}_0$$

$$\vec{\nabla}\times\vec{B} - \mu\varepsilon\frac{\partial\vec{E}}{\partial t} = 0 \Rightarrow \vec{k}\times\vec{B}_0 + \mu\varepsilon\,\omega\vec{E}_0 = 0$$

$$\Rightarrow \vec{E}_0 = -\frac{\omega}{\vec{k}^2}\vec{k}\times\vec{B}_0$$

$\Rightarrow \vec{k}, \vec{E}_0$ & $\vec{B}_0$ are orthogonal to each other! $|\vec{B}_0| = \sqrt{\mu\varepsilon}\,|\vec{E}_0|$



# Energy - Momentum Conservation in LIH Medium

energy density: $u = \dfrac{E^2 + B^2}{8\pi}$ ~ vacuum, Gaussian units

$$\Downarrow$$

$$u = \frac{\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}}{2}$$ ~ LIH medium, SI

Poynting vector: $\vec{S} = \dfrac{c}{4\pi}\,\vec{E}\times\vec{B}$ ~ vacuum, Gaussian units

$$\Downarrow$$

$\vec{P}_{field} = \displaystyle\int \frac{\vec{S}}{c^2}\, d^3x$ still. $\quad \vec{S} = \vec{E}\times\vec{H}$ ~ LIH, SI

Energy conservation: still have $\boxed{\dfrac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\underbrace{\vec{J}\cdot\vec{E}}_{-\frac{\partial u_{mech}}{\partial t}}}$

Maxwell stress tensor:

$$\sigma^{ij} = \frac{1}{4\pi}\left[E^iE^j + B^iB^j - \frac{1}{2}\delta^{ij}\left(E^2 + B^2\right)\right]$$ ~ vacuum, Gaussian units

$$\Downarrow$$

$$\sigma^{ij} = \varepsilon E^iE^j + \mu H^iH^j - \frac{1}{2}\delta^{ij}\left(\varepsilon E^2 + \mu H^2\right)$$ ~ LIH, SI.

$$\boxed{\frac{d}{dt}\left[P^i_{field} + P^i_{mech}\right] = \oint_S da\, n^j \sigma^{ij}}$$ momentum conservation remains the same

Energy density

$$u = \frac{1}{2}\left(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H}\right) = \frac{1}{2}\left(\varepsilon E^2 + \mu H^2\right) = \left[\frac{1}{2}\varepsilon E_0^2 + \frac{1}{2\mu}B_0^2\right]\cos^2(\vec{k}\cdot\vec{x} - \omega t) = \varepsilon E_0^2 \cos^2(\vec{k}\cdot\vec{x} - \omega t)$$

$\Rightarrow$ time averaged $\langle u \rangle = \varepsilon E_0^2 \underbrace{\langle \cos^2(\vec{k}\cdot\vec{x} - \omega t)\rangle}_{=1/2} =$

$= \frac{1}{2}\varepsilon E_0^2 \Rightarrow \langle u \rangle = \frac{1}{2}\varepsilon E_0^2$

Poynting vector

$$\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{1}{\mu}E_0 B_0 \hat{k}\cos^2(\vec{k}\cdot\vec{x} - \omega t) =$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\, E_0^2\, \hat{k}\cos^2(\vec{k}\cdot\vec{x} - \omega t)$$

$\Rightarrow$ time averaged $\langle \vec{S} \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\, E_0^2\, \hat{k}$ energy flow

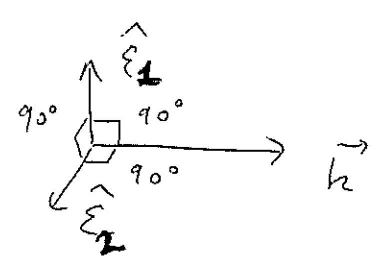
$\langle \vec{S} \rangle = \frac{1}{\sqrt{\mu\varepsilon}}\langle u \rangle \hat{k} \Rightarrow$ energy is traveling with velocity $\vec{v} = \frac{\hat{k}}{\sqrt{\mu\varepsilon}} = \frac{\omega}{k}\hat{k}$.

## Polarization.

Plane wave: $\vec{E} = \mathrm{Re}\left\{\vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}\right\}$, $\vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E}$

in the future drop Re sign.

Choose a basis: $\hat{\varepsilon}_1$ & $\hat{\varepsilon}_2$ in the plane transverse to $\vec{k}$



$$\Rightarrow \vec{E} = \left(\hat{\varepsilon}_1 E_1 + \hat{\varepsilon}_2 E_2\right) e^{i(k\cdot x - \omega t)}$$

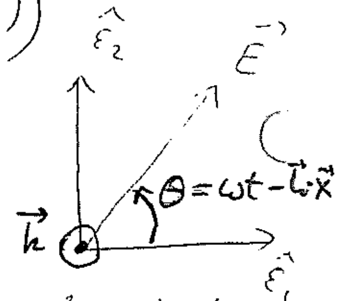
$E_1, E_2$ ~ are coefficients, generally complex.

Def. if $E_1$ & $E_2$ have the same phases the wave is <u>linearly polarized</u> in the direction defined by angle $\theta = \tan^{-1}\left(\frac{E_2}{E_1}\right)$, with amplitude $E = \sqrt{|E_1|^2 + |E_2|^2}$

Suppose phases of $E_1$ & $E_2$ differ by $90°$: $E_2 = \pm i E_1$

$$\Rightarrow \vec{E} = E_1 \left(\hat{\varepsilon}_1 \pm i \hat{\varepsilon}_2\right) e^{i(\vec{k}\cdot\vec{x} - \omega t)} = \text{taking Re} =$$

$$= E_1 \left( \hat{\varepsilon}_1 \cos(\omega t - \vec{k}\cdot\vec{x}) \pm \hat{\varepsilon}_2 \sin(\omega t - \vec{k}\cdot\vec{x})\right)$$




the wave direction rotates with time $t$ at any fixed location $\vec{x}$.

Def.
$\Rightarrow$ Such wave is called <u>circularly polarized</u>.

$\hat{\varepsilon}_1 + i\hat{\varepsilon}_2$ ~ counterclockwise ~ left circ. polar. ~ positive helicity
— clockwise ~ right —"— ~ negative helicity

Change the basis: $\hat{\varepsilon}_\pm = \frac{1}{\sqrt{2}}\left(\hat{\varepsilon}_1 \pm i \hat{\varepsilon}_2\right)$

$$\Rightarrow \vec{E} = \left(E_+ \hat{\varepsilon}_+ + E_- \hat{\varepsilon}_-\right) e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

left- & right- polarized waves

If $E_+ = \pm E_-$ $\Rightarrow$ get linearly polarized wave

Any linearly polarized wave is a superposition of 2 circularly polarized waves with opposite helicities.