Last time | Monochromatic Plane Wave

energy density: $u = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$

Poynting vector: $\vec{S} = \vec{E} \times \vec{H}$
(energy flow)

Maxwell stress tensor: $\sigma^{ij} = \varepsilon E^i E^j + \mu H^i H^j - \frac{1}{2}\delta^{ij}(\varepsilon E^2 + \mu H^2)$
(momentum flow)

(for LIH media)

$$\begin{cases} \vec{E} = \vec{E}_0 \cos(\omega t - \vec{k}\cdot\vec{x}) \\ \vec{B} = \vec{B}_0 \cos(\omega t - \vec{k}\cdot\vec{x}) \end{cases}$$
~ monochromatic plane wave

$$\langle u \rangle = \frac{1}{2}\varepsilon E_0^2$$

$$\langle \vec{S} \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\, E_0^2 \hat{k} = \langle u \rangle \vec{v}$$ ~ energy flow

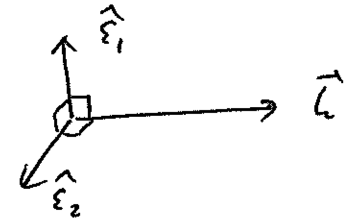
where $\vec{v} = \frac{\hat{k}}{\sqrt{\mu\varepsilon}}$ ~ phase velocity

$\langle \dots \rangle$ ~ time averaged

Polarization

$$\vec{E} = \mathrm{Re}\left\{ \vec{E}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \right\} \sim \text{plane wave}$$

choose a basis:



$$\hat{E}_0 = E_1\,\hat{\varepsilon}_1 + E_2\,\hat{\varepsilon}_2 .$$

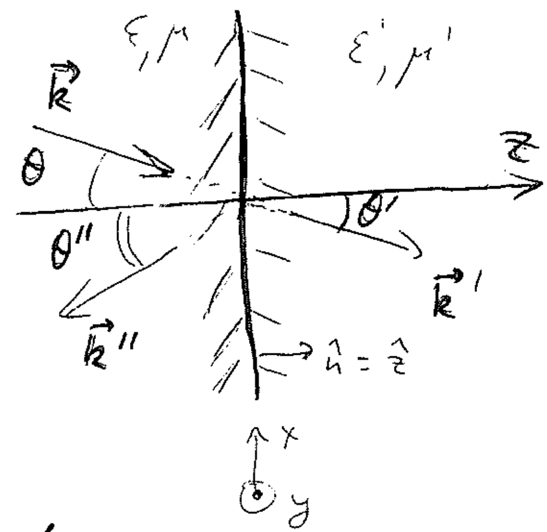
E_1 & E_2 are complex in general (may contain phase shifts)

$\hat{\varepsilon}_1, \hat{\varepsilon}_2 \sim$ linear polarizations

Def. $\hat{\varepsilon}_\pm = \frac{1}{\sqrt{2}}\left(\hat{\varepsilon}_1 \pm i\,\hat{\varepsilon}_2 \right)$ ~ circular polarizations (a different basis)

$$\Rightarrow \quad \vec{E}_0 = E_+\,\hat{\varepsilon}_+ + E_-\,\hat{\varepsilon}_- .$$

Reflection and Refraction



(do not assume that all $\vec{k}$'s are in one plane)

Incident wave:

$$\begin{cases} \vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)} \\ \vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E} = \sqrt{\mu\varepsilon} \, \frac{\vec{k} \times \vec{E}}{k} . \end{cases}$$

Refracted wave:

$$\begin{cases} \vec{E}' = \vec{E}_0' \, e^{i(\vec{k}'\cdot\vec{x} - \omega' t)} \\ \vec{B}' = \sqrt{\mu'\varepsilon'} \, \frac{\vec{k}' \times \vec{E}'}{k'} \end{cases}$$

Reflected wave:
$$\begin{cases} \vec{E}'' = \vec{E}_0'' \, e^{i(\vec{k}''\cdot\vec{x} - \omega'' t)} \\ \vec{B}'' = \sqrt{\mu\varepsilon} \, \frac{\vec{k}'' \times \vec{E}''}{k''} . \end{cases}$$

Match boundary conditions. $\vec{\nabla}\cdot\vec{D} = 0 \Rightarrow D_n$ is cont.

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow B_n$ is continuous

$\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \vec{\nabla}\times\vec{E} - i\omega\vec{B} = 0 \Rightarrow E_t$ is continuous

as $\vec{B}$ has no δ-function singularity at $z=0$

$\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} = -i\omega\vec{D} \Rightarrow H_t$ is continuous (same reason)

to have any boundary conditions at $z=0$ need

$$\omega = \omega' = \omega'' \Rightarrow k = k'' = \sqrt{\mu\varepsilon}\,\omega, \quad k' = \sqrt{\mu'\varepsilon'}\,\omega$$

spatial phase factors should also be equal

at $z=0$: $\vec{k}\cdot\vec{x}\Big|_{z=0} = \vec{k}'\cdot\vec{x}\Big|_{z=0} = \vec{k}''\cdot\vec{x}\Big|_{z=0}, \ \forall x, y$

$\Rightarrow$ choose $\vec{k} = (k_x, 0, k_z) \Rightarrow \vec{k}\cdot\vec{x}\big|_{z=0} = k_x \cdot x \Rightarrow$ no y-dep

$\Rightarrow$ there should be no y-dependence in $\vec{k}'\cdot\vec{x}$ and in $\vec{k}''\cdot\vec{x}$ as well $\Rightarrow$ $k_y' = k_y'' = 0$ $\Rightarrow$ all lie in the same plane

$$k \cdot \sin\theta = k' \cdot \sin\theta' = k'' \cdot \sin\theta''$$

$\Rightarrow$ as $k = k''$ $\Rightarrow$ $\theta = \theta''$ = angle of reflection is equal to angle of incidence!

as $k = \sqrt{\mu\varepsilon}\,\omega$ and $k' = \sqrt{\mu'\varepsilon'}\,\omega$ $\Rightarrow$

$\sqrt{\mu\varepsilon}\,\sin\theta = \sqrt{\mu'\varepsilon'}\,\sin\theta'$. Remember $n = c\sqrt{\mu\varepsilon}$ (index of refraction) $\Rightarrow$ $n\sin\theta = n'\sin\theta'$

Snell's law!

The only thing left is to find $\vec{E}_0'$ & $\vec{E}_0''$ using b.c.'s:

D_n $(= \varepsilon E_n)$ continuous $\Rightarrow$ $\hat{n}\cdot\left[\varepsilon(\vec{E}_0 + \vec{E}_0'') - \varepsilon'\vec{E}_0'\right] = 0$

B_n continuous $\Rightarrow$ $\hat{n}\cdot\left[\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0'' - \vec{k}'\times\vec{E}_0'\right] = 0$
(and $\omega = \omega' = \omega''$)

E_t continuous $\Rightarrow$ $\hat{n}\times\left[\vec{E}_0 + \vec{E}_0'' - \vec{E}_0'\right] = 0$

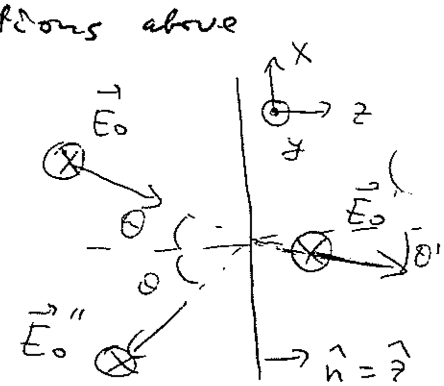
H_t continuous $\Rightarrow$ $\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0''\right) - \frac{1}{\mu'}\left(\vec{k}'\times\vec{E}_0'\right)\right]\times\hat{n} = 0$

$\vec{E}_0', \vec{E}_0''$ = 6 unknowns, while there are 6 equations above

Consider 2 cases: (linear polarization)

(I) $\vec{E}_0 \perp$ to the plane of incidence

$\vec{E}_0, \vec{E}_0', \vec{E}_0'' \parallel \hat{y}$



3rd & 4th equations ($\hat{n} = \hat{z}$)

$$\begin{cases} E_0 + E_0'' - E_0' = 0 \\ \frac{1}{\mu}\left(k E_0 \cos\theta - k'' E_0'' \cos\theta''\right) - \frac{1}{\mu'} k' E_0' \cos\theta' = 0 \end{cases}$$

as $k = k'' = \sqrt{\mu\varepsilon}\,\omega$, $k' = \sqrt{\mu'\varepsilon'}\,\omega$ $\Rightarrow$ and $\theta = \theta''$

$$\begin{cases} E_0 + E_0'' - E_0' = 0 \\ \sqrt{\frac{\varepsilon}{\mu}}\left(E_0 - E_0''\right)\cos\theta - \sqrt{\frac{\varepsilon'}{\mu'}} \cdot E_0' \cdot \cos\theta' = 0 \end{cases}$$

1st equ. $0 = 0$; 2nd equ.: $k E_0 \sin\theta + k'' E_0'' \sin\theta'' -$

$- k' E_0' \sin\theta' = 0 \Rightarrow \left(E_0 + E_0''\right)\sin\theta - \sqrt{\frac{\mu'\varepsilon'}{\mu\varepsilon}}\, E_0' \sin\theta' = 0$

as $\sqrt{\mu\varepsilon}\sin\theta = \sqrt{\mu'\varepsilon'}\sin\theta'$ (Snell's law) $\Rightarrow E_0 + E_0'' - E_0' = 0$

$\Rightarrow$ duplicates the 3rd one.

Using Snell's law ($n \sin\theta = n' \sin\theta'$) to get rid of n' we write (work it out yourself):

$$\boxed{\begin{aligned} \frac{E_0'}{E_0} &= \frac{2n\cos\theta}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}} \\ \frac{E_0''}{E_0} &= \frac{n\cos\theta - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}} \end{aligned}}$$

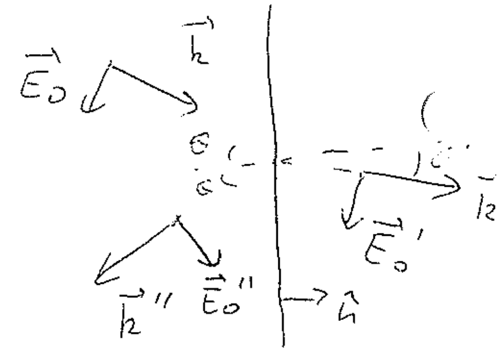
$n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$

$n' = \sqrt{\frac{\mu'\varepsilon'}{\mu_0\varepsilon_0}}$

Fresnel equations

(II) $\vec{E}_0 \parallel$ plane of incidence (xz plane)

2 independent equations (3rd & 4th):



$$\begin{cases} (E_0 + E_0'')\cos\theta - E_0'\cos\theta' = 0 \\ \sqrt{\dfrac{\varepsilon}{\mu}}\,(E_0 - E_0'') - \sqrt{\dfrac{\varepsilon'}{\mu'}}\,E_0' = 0 \end{cases}$$

(other two can be reduced to those)

Solve: (using Snell's Law)

$$\boxed{\begin{aligned} \frac{E_0'}{E_0} &= \frac{2nn'\cos\theta}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} \\ \frac{E_0''}{E_0} &= \frac{-\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} \end{aligned}}$$

Fresnel equations

Normal incidence: $\theta = 0 \Rightarrow$ both (I) and (II) give

$$\frac{E_0'}{E_0} = \frac{2n}{n + \frac{\mu}{\mu'}n'}, \qquad \frac{E_0''}{E_0} = \frac{n - \frac{\mu}{\mu'}n'}{n + \frac{\mu}{\mu'}n'}$$

if $\mu = \mu'$ & $n' > n$ $\Rightarrow$ reflected wave change sign - phase revers.

Polarization by reflection: put $\mu = \mu'$ for simplicity.

(I): $$\frac{E_0''}{E_0} = \frac{n\cos\theta - \sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \sqrt{n'^2 - n^2\sin^2\theta}}$$ different $\Rightarrow$

(II): $$\frac{E_0''}{E_0} = \frac{-n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}{n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}$$ $\Rightarrow$ reflected light is polarized!