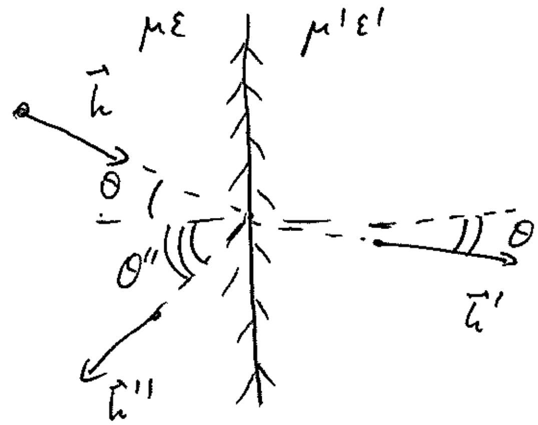
Last time | **Reflection and Refraction** (cont'd)

incoming wave

$$\begin{cases} \vec{E} = \vec{E}_0 e^{-i\omega t + i\vec{k}\cdot\vec{x}} \\ \vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E} \end{cases}$$



refracted wave

$$\begin{cases} \vec{E}' = \vec{E}_0' e^{-i\omega' t + i\vec{k}'\cdot\vec{x}} \\ \vec{B}' = \frac{1}{\omega'}\vec{k}'\times\vec{E}' \end{cases}$$

reflected wave

$$\begin{cases} \vec{E}'' = \vec{E}_0'' e^{-i\omega'' t + i\vec{k}''\cdot\vec{x}} \\ \vec{B}'' = \frac{1}{\omega''}\vec{k}''\times\vec{E}'' \end{cases}$$

Boundary conditions: D_n, B_n, H_t, E_t are continuous

Matching the t- and $\vec{x}$-dependent phases we get

$\boxed{\omega = \omega' = \omega''}$ $\quad k = k'' = \sqrt{\mu\varepsilon}\,\omega, \quad k' = \sqrt{\mu'\varepsilon'}\,\omega.$

$\boxed{\theta = \theta''}$ and Snell's Law $\boxed{n\sin\theta = n'\sin\theta'}$

angle of incidence = angle of reflection

Boundary conditions yield for $\vec{E}_0'$, $\vec{E}_0''$:

D_n: $\quad \hat{n}\cdot\left[\varepsilon(\vec{E}_0+\vec{E}_0'') - \varepsilon'\vec{E}_0'\right] = 0$

B_n: $\quad \hat{n}\cdot\left[\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0'' - \vec{k}'\times\vec{E}_0'\right] = 0$

E_t: $\quad \hat{n}\times\left[\vec{E}_0 + \vec{E}_0'' - \vec{E}_0'\right] = 0$

H_t: $\quad \hat{n}\times\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0''\right) - \frac{1}{\mu'}\vec{k}'\times\vec{E}_0'\right] = 0.$

3rd & 4th equations ($\hat{n} = \hat{z}$)

$$\begin{cases} E_0 + E_0'' - E_0' = 0 \\ \frac{1}{\mu}\left(k E_0 \cos\theta - k'' E_0'' \cos\theta''\right) - \frac{1}{\mu'} k' E_0' \cos\theta' = 0 \end{cases}$$

as $k = k'' = \sqrt{\mu\varepsilon}\ \omega$, $k' = \sqrt{\mu'\varepsilon'}\ \omega$ $\Rightarrow$ and $\theta = \theta''$

$$\begin{cases} E_0 + E_0'' - E_0' = 0 \\ \sqrt{\frac{\varepsilon}{\mu}}\left(E_0 - E_0''\right)\cos\theta - \sqrt{\frac{\varepsilon'}{\mu'}} \cdot E_0' \cdot \cos\theta' = 0 \end{cases}$$

1st equ. $0 = 0$, 2nd equ.: $k E_0 \sin\theta + k'' E_0'' \sin\theta'' -$
$- k' E_0' \sin\theta' = 0 \Rightarrow \left(E_0 + E_0''\right)\sin\theta - \sqrt{\frac{\mu'\varepsilon'}{\mu\varepsilon}}\, E_0' \sin\theta' = 0$

as $\sqrt{\mu\varepsilon}\sin\theta = \sqrt{\mu'\varepsilon'}\sin\theta'$ (Snell's law) $\Rightarrow$ $E_0 + E_0'' - E_0' = 0$

$\Rightarrow$ duplicates the 3rd one.

Using Snell's law ($n\sin\theta = n'\sin\theta'$) to get rid of n' we write (work it out yourself):

$$\boxed{\begin{aligned} \frac{E_0'}{E_0} &= \frac{2n\cos\theta}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}} \\ \frac{E_0''}{E_0} &= \frac{n\cos\theta - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}} \end{aligned}}$$

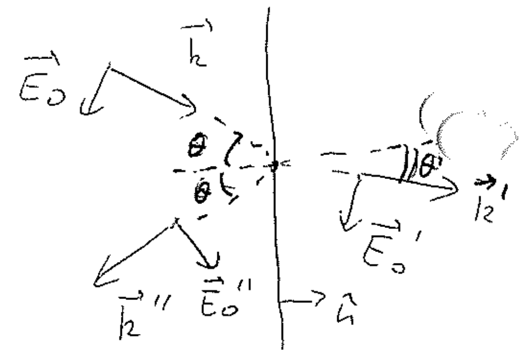
$n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$

$n' = \sqrt{\frac{\mu'\varepsilon'}{\mu_0\varepsilon_0}}$

Fresnel equations

(II) $\vec{E}_0 \parallel$ plane of incidence (xz plane)

2 independent equations (3rd & 4th):



$$\begin{cases} (E_0 + E_0'')\cos\theta - E_0'\cos\theta' = 0 \\ \sqrt{\frac{\varepsilon}{\mu}}\,(E_0 - E_0'') - \sqrt{\frac{\varepsilon'}{\mu'}}\,E_0' = 0 \end{cases}$$

(other two can be reduced to those)

Solve: (using Snell's Law)

$$\boxed{\begin{aligned} \frac{E_0'}{E_0} &= \frac{2nn'\cos\theta}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} \\ \frac{E_0''}{E_0} &= \frac{-\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}{\frac{\mu}{\mu'}n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} \end{aligned}}$$

Fresnel equations

Normal incidence: $\theta = 0 \Rightarrow$ both (I) and (II) give

$$\frac{E_0'}{E_0} = \frac{2n}{n + \frac{\mu}{\mu'}n'}, \qquad \frac{E_0''}{E_0} = \frac{n - \frac{\mu}{\mu'}n'}{n + \frac{\mu}{\mu'}n'}$$

if $\mu = \mu'$ & $n' > n$ $\Rightarrow$ reflected wave change sign - phase revers.

<u>Polarization by reflection</u>: put $\mu = \mu'$ for simplicity.

(I): $$\frac{E_0''}{E_0} = \frac{n\cos\theta - \sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \sqrt{n'^2 - n^2\sin^2\theta}}$$

different $\Rightarrow$ reflected light is polarized!

(II): $$\frac{E_0''}{E_0} = \frac{-n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}{n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}$$

in case I $\quad \dfrac{E_0''}{E_0}$ never vanishes (always < 0) if $n'>n$

in case II $\quad \dfrac{E_0''}{E_0} = 0$ for $\boxed{\theta_B = \tan^{-1}\left(\dfrac{n'}{n}\right)}$ Brewster's angle

$$\left[\begin{array}{l} n^2 n'^2 - n^4 \sin^2\theta = n'^4 \cos^2\theta \Rightarrow \\ \Rightarrow n^2 n'^2 (1+\tan^2\theta) - n^4 \tan^2\theta = n'^4 \Rightarrow n^2 n'^2 - n'^4 = n^2(n^2 - n'^2)\tan^2\theta \Rightarrow \tan^2\theta = \dfrac{n'^2}{n^2} \end{array}\right]$$

$\Rightarrow$ reflected light is <u>polarized</u>.

if $\theta = \theta_B \Rightarrow$ polarization is linear, $\perp$ to the plane of incidence

(fish in the ocean reflect light ~ squids with polarized vision can see them)

<u>Total internal reflection</u>: Snell's law:

$$n \sin\theta = n' \sin\theta' \leq n' \Rightarrow \theta \leq \sin^{-1}\left(\frac{n'}{n}\right) \Rightarrow$$

$\Rightarrow$ for $\boxed{\theta > \sin^{-1}\left(\dfrac{n'}{n}\right)}$ get total reflection

"evanescent wave" $\Leftrightarrow$ imaginary $\vec{k}$

$$k' = \sqrt{\mu'\varepsilon'}\,\omega \Rightarrow k'_x = -\sqrt{\mu'\varepsilon'}\,\omega \sin\theta' = -\sqrt{\mu_0\varepsilon_0}\,\omega\, n'\sin\theta' =$$

$$= -\frac{\omega}{c} n \sin\theta$$

$\quad '' \; n'\dfrac{\omega}{c}$

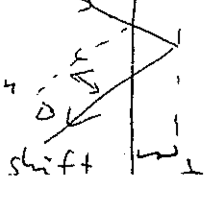
$$k'_y = 0$$

$$k'_z = \sqrt{k'^2 - k'^2_x} = \sqrt{n'^2 - n^2\sin^2\theta}\;\frac{\omega}{c}$$

$\Rightarrow$ for $\theta > \sin^{-1}\left(\dfrac{n'}{n}\right)$: k'_z becomes imaginary $k'_z = +i|k'_z|$

$$\Rightarrow e^{i k'_z z} \sim e^{-|k'_z| z} \sim \text{exponential falloff}$$

effectively the wave gets reflected from a different surface ~ violation of geom. optics, Goos-Hänchen effect



transmission coefficient $\vec{S} = \vec{E}\times\vec{H} = Re[\vec{E}\times\vec{H}^*]$

$$T = \frac{|\vec{S}'|}{|\vec{S}|} = \frac{E_0' \cdot H_0' \cdot \frac{1}{2}}{E_0 \cdot H_0 \cdot \frac{1}{2}} = \frac{\mu}{\mu'}\frac{E_0' \cdot B_0'}{E_0 B_0} = \frac{\mu}{\mu'}\frac{\sqrt{\mu'\varepsilon'}}{\sqrt{\mu\varepsilon}}\left(\frac{E_0'}{E_0}\right)^2$$

$\uparrow \langle \cos^2 \rangle$ ~ phase

if $\mu = \mu'$ $\quad T = \frac{4nn'}{(n+n')^2}$; $R = \left(\frac{n-n'}{n+n'}\right)^2$

$$= \Big|_{\theta=0}^{\text{for}} = \sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}} \cdot \frac{4n^2}{\left(n+\frac{\mu}{\mu'}n'\right)^2}$$

~ fraction of incident power that got through

reflection coefficient ~ fraction of inc. power reflected. $(T+R=1)$

$$R = \frac{|\vec{S}''|}{|\vec{S}|} = \frac{E_0'' H_0''}{E_0 H_0} = \frac{E_0'' B_0''}{E_0 B_0} = \frac{|E_0''|^2}{|E_0|^2} = \left(\frac{n-\frac{\mu}{\mu'}n'}{n+\frac{\mu}{\mu'}n'}\right)^2$$

Electromagnetic Waves in Conductors

maxwell equations: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$

$$\vec{\nabla}\cdot\vec{D} = \rho = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$\uparrow$ assume no net free charge

Ohm's law: $\vec{J} = \sigma\vec{E}$; $\vec{B} = \mu\vec{H}$, $\vec{D} = \varepsilon\vec{E}$

Look for plane-wave solutions:

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} \; ; \quad \vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = i\omega\vec{B} = i\omega\mu\vec{H} \qquad \vec{\nabla}\times\vec{H} = \underbrace{\sigma\vec{E} - i\omega\varepsilon\vec{E}}_{-i\omega\left(\varepsilon+\frac{i}{\omega}\sigma\right)}$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2\vec{E} = i\omega\mu\,\vec{\nabla}\times\vec{H} = i\omega\mu(\sigma - i\omega\varepsilon)\vec{E}$$

There are bound electrons giving ε_b, μ_b and free electrons giving $\vec{J}$.
Still $\rho=0$ due to charge neutrality. Note that $\vec{\nabla}\cdot\vec{J} = 0$.