Last time | **Kramers-Kronig Relations** (cont'd)

assumed that $\vec{D}(\vec{x},\omega) = \varepsilon(\omega)\vec{E}(\vec{x},\omega)$

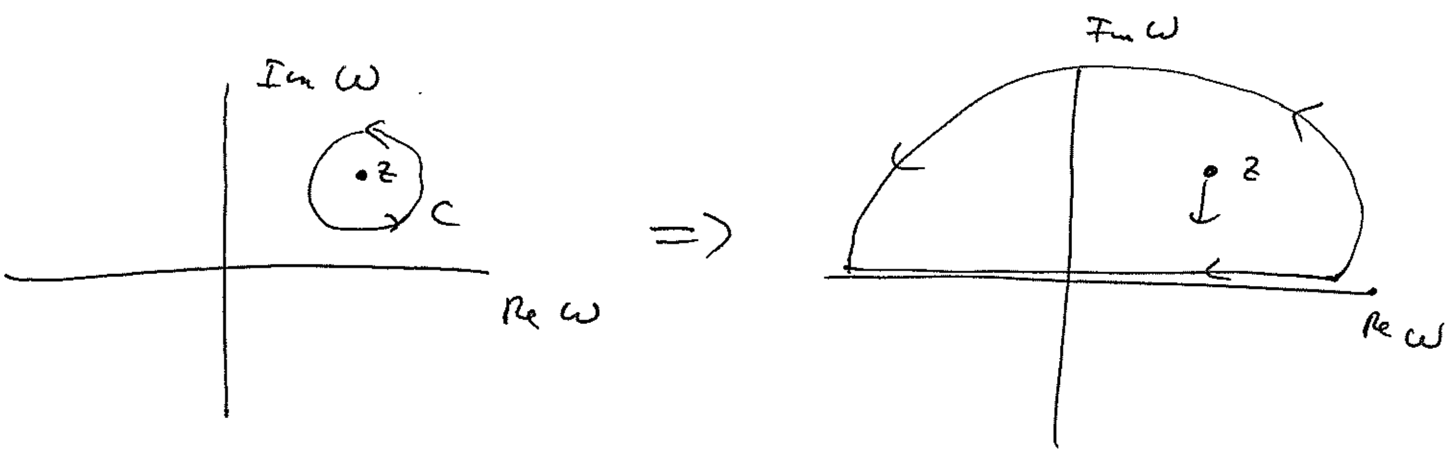
$$\Rightarrow \boxed{\vec{D}(\vec{x},t) = \varepsilon_0\left[\vec{E}(\vec{x},t) + \int_{-\infty}^{\infty} d\tau\, G(\tau)\,\vec{E}(\vec{x},t-\tau)\right]}$$

where $\boxed{G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\tau}\left[\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right].}$

$G(\tau) = 0$ for $\tau < 0$ $\Rightarrow$ causality.

$$\boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_0^{\infty} d\tau\, e^{i\omega\tau}\, G(\tau)}$$

$$\left.\begin{array}{l} \bullet\ G(\tau) \text{ is finite} \\ \bullet\ G(\tau) \to 0 \text{ as } \tau \to \infty \end{array}\right\} \Rightarrow \begin{array}{l} \varepsilon(\omega) \text{ is analytic} \\ \text{for } \operatorname{Im}\omega \geq 0. \end{array}$$



& drop the contribution of the semi-circle ⇒

$\Rightarrow$ we obtained

$$\boxed{\begin{aligned} \mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} &= 1 + \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \left(PV\,\frac{1}{\omega'-\omega}\right) \mathrm{Im}\,\frac{\varepsilon(\omega')}{\varepsilon_0} \\ \mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} &= -\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \left[\mathrm{Re}\,\frac{\varepsilon(\omega')}{\varepsilon_0} - 1\right]\left(PV\,\frac{1}{\omega'-\omega}\right) \end{aligned}}$$

Kramers - Kronig relations.

$$= \int_{-\infty}^{\infty} dt' \, \vec{E}(\vec{x},t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{i\omega(t'-t)} \left[\varepsilon(\omega) - \varepsilon_0 + \varepsilon_0 \right] =\Big/ \tau = t-t'$$

$$= \varepsilon_0 \vec{E}(\vec{x},t) + \varepsilon_0 \int_{-\infty}^{\infty} d\tau \; G(\tau) \, \vec{E}(\vec{x}, t-\tau)$$

such that $\vec{D}(\vec{x},t) = \varepsilon_0 \left\{ \vec{E}(\vec{x},t) + \int_{-\infty}^{\infty} d\tau \, G(\tau) \, \vec{E}(\vec{x}, t-\tau) \right\}$

linear response

with $\boxed{G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega\tau} \left[\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right]}$

Usually $G(\tau) = 0$ for $\tau < 0$ => causality : $\vec{D}(\vec{x},t)$ is affected by $\vec{E}(\vec{x},t)$ (instantaneous term) and by $\vec{E}(\vec{x},t')$ with $t' < t$ ~ **delayed action**.

<u>Example:</u> in a simple model above we had :

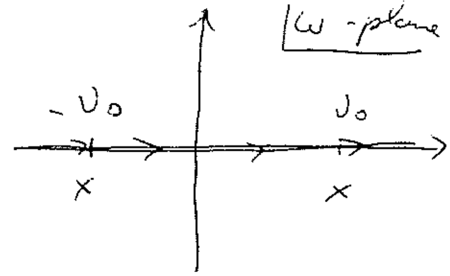
$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma_0 - \omega^2}$$

$$\Rightarrow G(\tau) = \omega_p^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega\tau} \frac{1}{\omega_0^2 - i\omega\gamma_0 - \omega^2}$$

poles at $\omega^2 + i\omega\gamma_0 - \omega_0^2 = 0$

$$\omega_{1,2} = \frac{1}{2}\left[-i\gamma_0 \pm \sqrt{-\gamma_0^2 + 4\omega_0^2} \right] = \pm \sqrt{\underbrace{\omega_0^2 - \frac{\gamma_0^2}{4}}_{\nu_0^2}} - \frac{i\gamma_0}{2} =$$

$$= \pm \nu_0 - i\frac{\gamma_0}{2}$$



$$\Rightarrow G(\tau) = -\omega_p^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \cdot \frac{1}{\left(\omega - \nu_0 + \frac{i\gamma_0}{2}\right)\left(\omega + \nu_0 + \frac{i\gamma_0}{2}\right)} = -\omega_p^2 \cdot \theta(\tau) \cdot (-2\pi i) \cdot \frac{1}{2\pi}$$

$$\cdot \left[\frac{1}{2\nu_0} e^{-i\left(\nu_0 - i\frac{\gamma_0}{2}\right)\tau} + \frac{1}{-2\nu_0} e^{+i\left(\nu_0 + \frac{i\gamma_0}{2}\right)\tau} \right] =$$

$$= \frac{\omega_p^2}{2\nu_0} \theta(\tau) \cdot i \cdot (-2i) \sin(\nu_0 \tau)\, e^{-\frac{\gamma_0}{2}\tau}$$

$$\Rightarrow G(\tau) = \theta(\tau)\, \omega_p^2\, e^{-\frac{\gamma_0 \tau}{2}} \frac{\sin(\nu_0 \tau)}{\nu_0}$$

$G(\tau) \sim \theta(\tau) \sim$ causality

$G(\tau) \sim e^{-\frac{\gamma_0}{2}\tau} \sim$ you can feel the effects from back in time only so much.

Invert the expression for $G(\tau)$: first, assuming that $G(\tau) = 0$ for $\tau < 0$ write:

$$\vec{D}(\vec{x}, t) = \varepsilon_0 \left\{ \vec{E}(\vec{x}, t) + \int_0^{\infty} d\tau\, G(\tau)\, \vec{E}(\vec{x}, t - \tau) \right\}$$

$$\Rightarrow G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \left[\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right] \Rightarrow$$

$$\Rightarrow \int_0^{\infty} d\tau\, e^{i\omega\tau} G(\tau) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} G(\tau) = \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \Rightarrow$$

$$\Rightarrow \boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_0^\infty d\tau\, e^{i\omega\tau}\, G(\tau)}$$

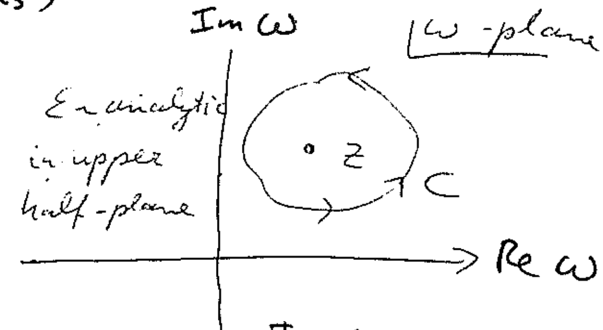
$\mathrm{Im}\,\varepsilon(-\omega) = -\mathrm{Im}\,\varepsilon(\omega)$
$\mathrm{Re}\,\varepsilon(-\omega) = \mathrm{Re}\,\varepsilon(\omega)$
$\Uparrow$

$\vec{E}$, $\vec{D}$ are real $\Rightarrow$ G is real $\Rightarrow$ $\dfrac{\varepsilon(-\omega)}{\varepsilon_0} = \dfrac{\varepsilon^*(\omega^*)}{\varepsilon_0}$

Physically reasonable $G(\tau) \to 0$ as $\tau \to \infty$ & if $G(\tau)$ is finite $\Rightarrow$ $\varepsilon(\omega)$ is analytic for $\mathrm{Im}\,\omega \geq 0$. (e.g. see the retarded Green fn. calculation)
(including $\mathrm{Im}\,\omega = 0$ ~ real axis)

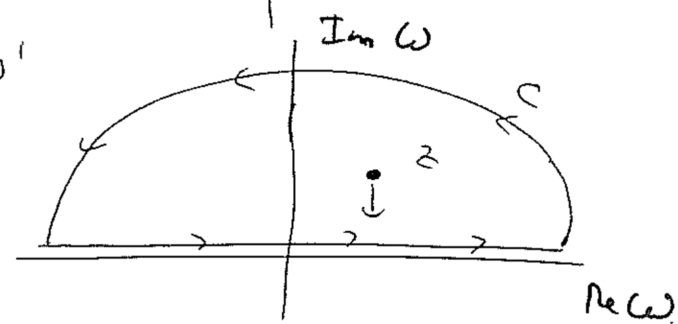
$G(0) \equiv 0$ ~ continuity!
(take $\tau \to +0$)

Use Cauchy's theorem:



$$\frac{\varepsilon(z)}{\varepsilon_0} = 1 + \frac{1}{2\pi i} \oint_C \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - z}\, d\omega'$$

Distort C-contour to $\rightarrow$



and take $\mathrm{Im}\, z \to +0$.

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_0^\infty d\tau\, \underbrace{e^{i\omega\tau}}_{= \frac{-i}{\omega}\frac{d}{d\tau} e^{i\omega\tau}}\, G(\tau) = 1 - \frac{i}{\omega}\int_0^\infty d\tau\, G(\tau) \frac{d}{d\tau} e^{i\omega\tau} =$$

$$= (\text{parts}) = 1 - \frac{i}{\omega} \underbrace{G(\tau) e^{i\omega\tau}\Big|_0^\infty}_{0} + \frac{i}{\omega}\int_0^\infty d\tau\, e^{i\omega\tau} G'(\tau) =$$

$$= \begin{pmatrix}\text{parts}\\ \text{again}\end{pmatrix} = 1 + \frac{e^{i\omega\tau}}{\omega^2} G'(\tau)\Big|_0^\infty = -G'(0)/\omega^2 - \frac{1}{\omega^2}\int_0^\infty d\tau\, e^{i\omega\tau} G''(\tau) = o\left(\frac{1}{\omega^2}\right)$$

$\Rightarrow$ neglect the semi-circle part of contour.

$\Rightarrow \mathrm{Re}\left(\frac{\varepsilon(\omega)}{\varepsilon} - 1\right) \sim \frac{1}{\omega^2}$, $\mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon} \sim \frac{1}{\omega^3}$. as $\omega \to \infty$.

Write $z = \omega + i\delta$, $\omega \sim$ real

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - \omega - i\delta}$$

use
$$P\frac{1}{x} = \frac{1}{2}\left(\frac{1}{x+i\varepsilon} + \frac{1}{x-i\varepsilon}\right).$$
$$\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = 2\pi i\,\delta(x)$$

as $$\frac{1}{\omega' - \omega - i\delta} = P\left(\frac{1}{\omega'-\omega}\right) + \pi i\,\delta(\omega' - \omega) \Rightarrow$$

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{2}\left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right) + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} P\left(\frac{1}{\omega'-\omega}\right)\left[\frac{\varepsilon(\omega')}{\varepsilon_0} - 1\right]$$

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{\pi i} P\int_{-\infty}^{\infty} d\omega' \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - \omega}$$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} &= 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Im}\left(\varepsilon(\omega')/\varepsilon_0\right)}{\omega' - \omega} \\ \mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} &= -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathrm{Re}\left(\varepsilon(\omega')/\omega_0\right) - 1}{\omega' - \omega}\, d\omega' \end{aligned}}$$

Kramers - Kronig relations. '26 - '27

If you know $\mathrm{Im}\,\varepsilon(\omega) \Rightarrow$ can find $\mathrm{Re}\,\varepsilon(\omega)$

& vice versa.

as $\mathrm{Re}\,\varepsilon(\omega) \sim \frac{1}{\omega^2}$ as $\omega \to \infty \Rightarrow$ define plasma frequency

as $$\omega_p^2 \equiv \lim_{\omega\to\infty}\left\{\omega^2\left[1 - \frac{\varepsilon(\omega)}{\varepsilon_0}\right]\right\} \Rightarrow \omega_p^2 = \frac{2}{\pi}\int_0^{\infty} d\omega\cdot\omega\cdot \mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0}.$$

as $\varepsilon(-\omega) = \varepsilon^*(\omega^*)$ => writing $\varepsilon(\omega) = \varepsilon_1(\omega) + i\,\varepsilon_2(\omega)$

we get $\varepsilon_1(-\omega) + i\,\varepsilon_2(-\omega) = \varepsilon_1(\omega) - i\,\varepsilon_2(\omega)$

along the real axis => $\boxed{\begin{aligned} \mathrm{Re}\,\varepsilon(\omega) &= \mathrm{Re}\,\varepsilon(-\omega) \\ \mathrm{Im}\,\varepsilon(\omega) &= -\mathrm{Im}\,\varepsilon(-\omega) \end{aligned}}$

=> Kramers - Kronig relations can be re-written as

$$\mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{\pi}\left\{ PV\int_0^\infty d\omega' \,\frac{\mathrm{Im}\,\frac{\varepsilon(\omega')}{\varepsilon_0}}{\omega'-\omega} + \int_{-\infty}^{0} d\omega' \cdot \mathrm{Im}\,\frac{\varepsilon(\omega')}{\varepsilon_0} \cdot PV\,\frac{1}{\omega'-\omega} \right\} = 1 + \frac{1}{\pi}\int_0^\infty d\omega' \,\mathrm{Im}\,\frac{\varepsilon(\omega')}{\varepsilon_0}\left[PV\,\frac{1}{\omega'-\omega} + PV\,\frac{1}{\omega'+\omega}\right]$$

$$\Rightarrow \boxed{\mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{2}{\pi}\int_0^\infty d\omega' \,\mathrm{Im}\,\frac{\varepsilon(\omega')}{\varepsilon_0}\, PV\,\frac{\omega'}{\omega'^2-\omega^2}}$$

Similarly one can show that

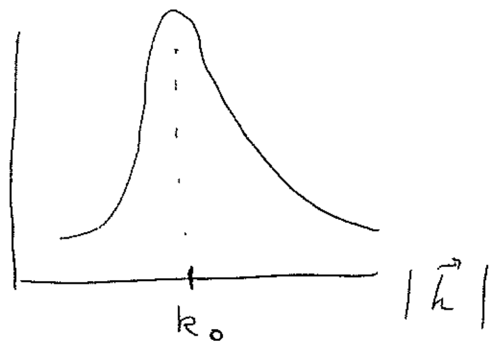
$$\boxed{\mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = -\frac{2\omega}{\pi}\int_0^\infty d\omega' \left[\mathrm{Re}\,\frac{\varepsilon(\omega')}{\varepsilon_0} - 1\right] PV\left(\frac{1}{\omega'^2-\omega^2}\right)}.$$

Group and Phase Velocities.

Consider a wave packet:

$$\vec{E}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, \vec{E}_0(\vec{k})\, e^{i\left(\vec{k}\cdot\vec{x} - \omega t\right)}$$

with $E_0(\vec{k})$



$$\Rightarrow \omega(\vec{k}) \approx \omega_0 + (\vec{k} - \vec{k}_0)\cdot(\vec{\nabla}_k \omega)\Big|_{\vec{k}=\vec{k}_0}$$

$$\Rightarrow \vec{k}\cdot\vec{x} - \omega t = \vec{k}\cdot\vec{x} - \omega_0 t - t(\vec{k} - \vec{k}_0)\cdot(\vec{\nabla}_k \omega)\Big|_{\vec{k}=\vec{k}_0} =$$

$$= \vec{k}\cdot\left(\vec{x} - t(\vec{\nabla}_k \omega)\Big|_{\vec{k}=\vec{k}_0}\right) \underbrace{- \omega_0 t + t\,\vec{k}_0\cdot(\vec{\nabla}_k \omega)\Big|_{\vec{k}=\vec{k}_0}}_{\text{overall factor (phase)}}$$

$$\Rightarrow \boxed{\vec{V}_g = \vec{\nabla}_k \omega\Big|_{\vec{k}=\vec{k}_0}}$$ group velocity ~

as $\vec{E}(\vec{x},t) = \underbrace{e^{-i\omega_0 t + i t \vec{k}_0\cdot\vec{\nabla}_k\omega}}_{\text{overall phase } (\vec{k}\text{-indep.})} \vec{E}(\underbrace{\vec{x} - t\vec{\nabla}_{k_0}\omega}_{\text{traveling wave}}, 0)$

~ speed of the wave packet

cf. $V_{ph} = \dfrac{\omega}{k}$ ~ phase velocity.

$\Rightarrow$ energy is transferred with V_g, not V_{ph}

Example: $\dfrac{\varepsilon}{\varepsilon_0} = 1 - \dfrac{\omega_p^2}{\omega^2} \Rightarrow k = \omega\sqrt{\varepsilon\mu_0} = \omega\sqrt{\varepsilon_0\mu_0}\,.$

$$\sqrt{1 - \omega_p^2/\omega^2} = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2} \Rightarrow V_g = \frac{d\omega}{dk}\Big|_{k=k_0} =$$

$$= \frac{c}{d\sqrt{\omega^2-\omega_p^2}/d\omega} = c\,\frac{\sqrt{\omega^2-\omega_p^2}}{\omega} = c\sqrt{1-\frac{\omega_p^2}{\omega^2}} < c$$

$(\omega > \omega_p)$

$V_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1-\omega_p^2/\omega^2}} > c$ violation of relativity?

No, nothing really moves with V_{ph}, all physical quantities move with V_g!

Waveguides and Resonant Cavities

Waves propagating in confined spaces:

cavity ~ confined in all directions

waveguide ~ confined in all but one direction ~ extended object

Consider a perfect conductor: $\sigma \to \infty$.

(skin depth $\delta \sim \frac{1}{\sqrt{\sigma}} \to 0$).

$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$ and $\vec{J} = \sigma \vec{E} \Rightarrow \vec{E} = 0$ inside (otherwise $\vec{J}$ infinite)

$\vec{E} = 0 \Rightarrow \rho = 0$ inside $\Rightarrow$ can only have surface density Σ $\Rightarrow$ $\hat{n}\cdot\vec{D} = \Sigma$

as $\vec{E} = 0$ inside $\Rightarrow$ $\hat{n}\times\vec{E} = 0$

(boundary conditions)