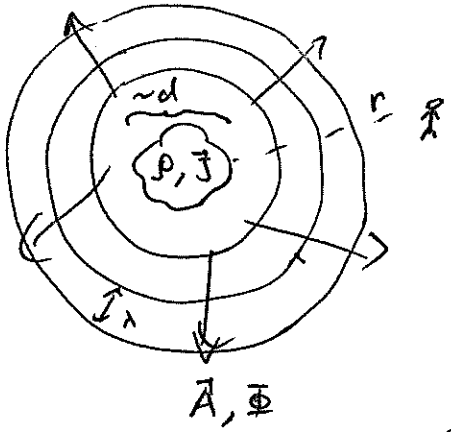
Last time

Radiation (cont'd)

Radiation by Harmonically Oscillating Sources



$$\begin{cases} \rho(\vec{x},t) = \rho(\vec{x})\, e^{-i\omega t} \\ \vec{J}(\vec{x},t) = \vec{J}(\vec{x})\, e^{-i\omega t} \end{cases}$$ localized charges

Solved Maxwell equations:

$k = \frac{\omega}{c}$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} e^{ik|\vec{x}-\vec{x}'|}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} \qquad \vec{E} = \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{H}$$

$\Rightarrow$ Far (radiation) zone: $d \ll \lambda \ll r$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \, \vec{J}(\vec{x}')\, e^{-ik\hat{n}\cdot\vec{x}'} \quad , \quad \hat{n} = \frac{\vec{x}}{|\vec{x}|}.$$

Expand in a series:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{m=0}^{\infty}\frac{(-ik)^m}{m!}\int d^3x' \, \vec{J}(\vec{x}')\,(\hat{n}\cdot\vec{x}')^m$$

Electric Dipole Radiation (cont'd)

keep $m=0$ term only $\Rightarrow$ we obtained

$$\boxed{\vec{A}(\vec{x}) = -\frac{i\mu_0\omega}{4\pi}\vec{p}\,\frac{e^{ikr}}{r}}$$

, where $\vec{p} = \int d^3x\,\vec{x}\,\rho(\vec{x})$ is the electric dipole moment (amplitude)

$$\Rightarrow \boxed{\vec{H} \simeq \frac{ck^2}{4\pi}\hat{n}\times\vec{p}\,\frac{e^{ikr}}{r}}$$

$$\boxed{\vec{E} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\vec{H}\times\hat{n}}$$

$$= \frac{i\omega}{4\pi} \vec{p} \times \hat{n} \cdot \left[ik \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right] \Rightarrow \text{as } \omega = ck$$

$$\hat{n} = \frac{\vec{r}}{r} = \hat{r}$$

$$\vec{H} = \frac{ck^2}{4\pi} \hat{n} \times \vec{p} \, \frac{e^{ikr}}{r} \left[1 - \frac{1}{ikr} \right]$$

$$\vec{E} = \frac{i}{\omega \varepsilon_0} \vec{\nabla} \times \vec{H} \Rightarrow \text{if } kr \gg 1 \Rightarrow$$

$$\Rightarrow \boxed{\vec{H} \approx \frac{ck^2}{4\pi} \hat{n} \times \vec{p} \, \frac{e^{ikr}}{r}} \Rightarrow \vec{E} = \frac{-i}{\varepsilon_0 \cancel{c}\cancel{k}} \frac{\cancel{c}k^2}{4\pi} (\hat{n} \times \vec{p}) \times \vec{\nabla} \cdot$$

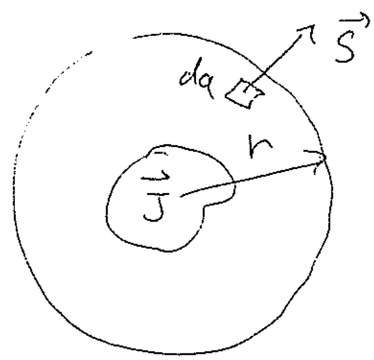
$$\frac{e^{ikr}}{r} \approx \frac{ik}{4\pi \varepsilon_0} ik (\hat{n} \times \vec{p}) \times \hat{n} \, \frac{e^{ikr}}{r} = \frac{1}{\varepsilon_0 c} \vec{H} \times \hat{n}$$

$$\Rightarrow \boxed{\vec{E} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \vec{H} \times \hat{n}}$$

$\vec{E} \perp \vec{H} \perp \hat{n}$ transverse

but only at large r.

Radiated power:



$$\frac{dP}{r^2 d\Omega} = \frac{dP}{da} = \hat{n} \cdot \vec{S}$$

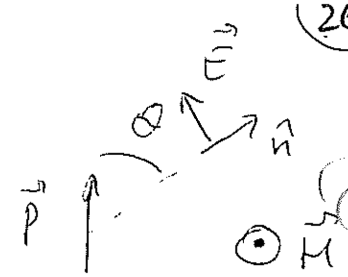
↖ unit of area

$$\Rightarrow \frac{dP}{d\Omega} = r^2 \hat{n} \cdot \vec{S} = \frac{1}{2} \mathrm{Re} \left[r^2 \hat{n} \cdot (\vec{E} \times \vec{H}^*) \right] =$$

↖ time-averaged!

$$= \frac{1}{2} r^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} |\vec{H}|^2 = \frac{1}{2} \cancel{r^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{c^2 k^4}{(4\pi)^2} \left| \hat{n} \times \vec{p} \right|^2 \frac{1}{\cancel{r^2}}$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4\, |\hat{n}\times\vec{p}|^2$$



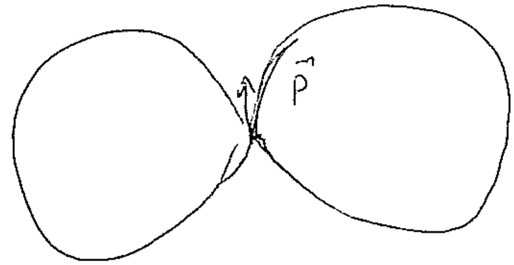
if all components of $\vec{p}$ have the same phase!

$$\Rightarrow \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 |\vec{p}|^2 \cdot \sin^2\theta.$$

Total emitted power: $P = \int d\Omega \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 |\vec{p}|^2.$

$$\cdot\, 2\pi \cdot \underbrace{\int_{-1}^{1} d\cos\theta\, (1-\cos^2\theta)}_{2-\frac{2}{3}\,=\,\frac{4}{3}} = \frac{c^2}{12\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 |\vec{p}|^2 = P$$

Radiation pattern:



Examples of dipoles:

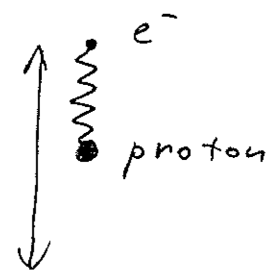
harmonically oscillating point charge on a spring

(i.e. "electron" in an "atom")

Example Atom as a harmonic oscillator: (hydrogen)

the position of electron is

$$\vec{X}(t) = \hat{z}\, r_0 \cos(\omega t) = \hat{z}\, \mathrm{Re}\left[r_0 e^{-i\omega t}\right]$$



$$\Rightarrow \vec{p} = -e\,\vec{X}(t) = -e\,\hat{z}\, r_0\, \mathrm{Re}\, e^{-i\omega t}$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4\, |\vec{p}|^2 \sin^2\theta = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 \cdot$$

$$\cdot\, e^2 r_0^2 \sin^2\theta$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{1}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, \frac{\omega^4}{c^2}\, e^2 r_0^2 \sin^2\theta$$

$$\Rightarrow P = \frac{1}{12\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, \frac{\omega^4}{c^2}\, |\vec{p}|^2 = \frac{1}{12\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, \frac{\omega^4}{c^2}\, e^2 r_0^2 = P$$

total radiated power.

~ Such classical atom would radiate energy

$\Rightarrow$ the electron would eventually fall on the proton $\Rightarrow$ need quantum mechanics to save the day!

Magnetic Dipole and Electric Quadrupole

Take the $n=1$ term:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int d^3x'\,\vec{J}(\vec{x}')\,\hat{n}\cdot\vec{x}'$$

$$\vec{J}(\hat{n}\cdot\vec{x}') = \frac{1}{2}\left[(\hat{n}\cdot\vec{x}')\vec{J} + (\hat{n}\cdot\vec{J})\vec{x}'\right] + \frac{1}{2}(\vec{x}'\times\vec{J})\times\hat{n}$$

as $J_i x_j = \frac{1}{2}(J_i x_j + J_j x_i) + \frac{1}{2}(J_i x_j - J_j x_i)$

(a) take the 2nd term first: remember the magnetization

$\vec{M} = \frac{1}{2}\vec{x}\times\vec{J} \Rightarrow$ the 2nd term gives

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot\int d^3x'\,\hat{n}\times\vec{M}(\vec{x}') \Rightarrow$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot\hat{n}\times\vec{m}$$ ~ magnetic dipole radiation.

where $\vec{m}$ is the magnetic dipole moment:

$$\frac{dP}{d\Omega} = \frac{k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}|\hat{n}\times\vec{m}|^2$$

$$\frac{1}{2}\int d^3x\,\vec{x}\times\vec{J} = \vec{m} = \int d^3x'\,\vec{M}(\vec{x}')$$

$\Rightarrow$ can find $\vec{E}, \vec{H}$: $\vec{H} = \frac{k^2}{4\pi}\frac{e^{ikr}}{r}(\hat{n}\times\vec{m})\times\hat{n}$; $\vec{E} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}k^2\,\vec{m}\times\hat{n}\frac{e^{ikr}}{r}$

(b) take the 1st term: $\frac{1}{2}\int d^3x'\left[n_i x_i' J_j + n_i J_i x_j'\right] =$

$$= \frac{1}{2}\int d^3x'\,\vec{J}\cdot\vec{\nabla}'\left(x_j'(\hat{n}\cdot\vec{x}')\right) = (\text{parts}) = -\frac{1}{2}\int d^3x'\,x_j'(\hat{n}\cdot\vec{x}')$$

$$\cdot\underbrace{\vec{\nabla}'\cdot\vec{J}}_{i\omega\rho} = -\frac{i\omega}{2}\int d^3x'\,x_j'(\hat{n}\cdot\vec{x}')\rho(\vec{x}') \Rightarrow$$