Last time | Magnetic Dipole and Electric Quadrupole

$$Q_{ij} = \int d^3x\, \rho(\vec{x}) \left[3x_i x_j - r^2 \delta_{ij} \right]$$

$$\vec{Q}^{\,i} = Q^{ij} n^j$$

$$\vec{H} = -\frac{i\omega k^2}{24\pi} \frac{e^{ikr}}{r} \hat{n} \times \vec{Q}$$

$$\vec{E} = -\frac{1}{c\varepsilon_0} \hat{n} \times \vec{H}$$

angular distribution of radiation

$$\frac{dP}{d\Omega} = \frac{c k^6}{2(24\pi)^2 \varepsilon_0} \left[Q_{ij} n_j Q_{ik}^* n_k - |Q_{ij} n_i n_j|^2 \right]$$

$$P = \frac{c k^6}{1440\pi \varepsilon_0} (Q_{ij})^2$$

~ net radiated power

$$\Rightarrow \vec{A}(\vec{x}) = -\frac{\mu_0}{4\pi}\frac{\omega k}{2}\frac{e^{ikr}}{r}\cdot\int d^3x'\cdot \vec{x}'\,(\hat{n}\cdot\vec{x}')\rho(\vec{x}')$$

electric quadrupole radiation

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} \simeq \frac{ik}{\mu_0}\hat{n}\times\vec{A}\,;\quad \vec{E} = \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{H} =$$

$$\simeq \frac{i}{\omega\varepsilon_0}\,ik\,\hat{n}\times\vec{H} = -\frac{1}{c\varepsilon_0}\frac{ik}{\mu_0}\,\hat{n}\times(\hat{n}\times\vec{A}) = -i\omega\,\hat{n}\times(\hat{n}\times\vec{A})$$

To find $\vec{H}$ need $\hat{n}\times\int d^3x'\,\vec{x}'\,(\hat{n}\cdot\vec{x}')\rho(\vec{x}')$.

Quadrupole moment tensor $Q_{ij} = \int d^3x\,(3x_ix_j - r^2\delta_{ij})\rho$

$$\Rightarrow \left[\hat{n}\times\int d^3x'\,\vec{x}'\,(\hat{n}\cdot\vec{x}')\rho(\vec{x}')\right]_i = \varepsilon_{ijk}\,n_j\int d^3x'\cdot x'_k\cdot$$

$$\cdot\, n_l\cdot x'_l\;\rho(\vec{x}') = \frac{1}{3}\varepsilon_{ijk}\,n_j\,n_l\,Q_{kl} = \frac{1}{3}\hat{n}\times\vec{Q}$$

with $(\vec{Q})_i = Q_{ij}\,n_j \Rightarrow$ $\vec{H} = -\frac{i}{8\pi}\omega k^2\frac{e^{ikr}}{r}\cdot\frac{1}{3}\hat{n}\times\vec{Q}$

$$\vec{E} = -\frac{1}{c\varepsilon_0}\hat{n}\times\vec{H} = \frac{i}{24\pi}\frac{\omega k^2}{c\varepsilon_0}\frac{e^{ikr}}{r}\,\hat{n}\times(\hat{n}\times\vec{Q})$$

Radiated power is (time averaged)

$$\frac{dP}{d\Omega} = \frac{1}{2}\operatorname{Re}\left[r^2\,\hat{n}\cdot(\vec{E}\times\vec{H}^*)\right] = -\frac{1}{2}\cancel{r^2}\frac{1}{3^2(8\pi)^2}\frac{\omega^2k^4}{c\varepsilon_0}\frac{1}{\cancel{r^2}}\cdot$$

$$\cdot\,\hat{n}\cdot\left((\hat{n}\times(\hat{n}\times\vec{Q}))\times(\hat{n}\times\vec{Q}^*)\right)$$

$$\vec{E} = -\frac{1}{c\varepsilon_0}\,\hat{n}\times\vec{H} \Rightarrow \frac{dP}{d\Omega} = \frac{r^2}{2}\,\mathrm{Re}\left[\hat{n}\cdot\left(\vec{E}\times\vec{H}^*\right)\right] = \frac{r^2}{2}\left(\frac{-1}{c\varepsilon_0}\right)$$

$$\mathrm{Re}\left[\hat{n}\cdot\left(\left(\hat{n}\times\vec{H}\right)\times\vec{H}^*\right)\right] = \frac{r^2}{2c\varepsilon_0}\,|\vec{H}|^2 = \frac{\cancel{r^2}}{2c\varepsilon_0}\cdot\frac{1}{64\pi^2}\,\omega^2 k^4\,\frac{1}{\cancel{r^2}}\,\frac{1}{9}$$

$$-\vec{H}^*\times(\hat{n}\times\vec{H}) = -\hat{n}\,|\vec{H}|^2 + \vec{H}\underbrace{\left(\hat{n}\cdot\vec{H}^*\right)}_{=0} = -\hat{n}\,|\vec{H}|^2$$

$$\cdot\,|\hat{n}\times\vec{Q}|^2 = \frac{\omega^2 k^4}{18\,c\varepsilon_0\,64\pi^2}\,|\hat{n}\times\vec{Q}|^2$$

$$\delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}$$

$$|\hat{n}\times\vec{Q}|^2 = \left|\varepsilon_{ijk}\,n_j\,Q_{kl}\,n_l\right|^2 = \varepsilon_{ijk}\,n_j\,Q_{kl}\,n_l\cdot\varepsilon_{ij'k'}\cdot n_{j'}\,Q^*_{k'l'}\,n_{l'} = n_j Q_{kl} n_l\, n_j Q^*_{kl'} n_{l'} - n_j Q_{kl} n_l\, n_k Q^*_{jl'} n_{l'}$$

$$= Q_{kl}\,n_l\,Q^*_{kl'}\,n_{l'} - Q_{kl}\,n_k n_l\,Q^*_{ij}\,n_i n_j$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{ck^6}{2(24\pi)^2\varepsilon_0}\left[Q_{ij}\,n_j\,Q^*_{ik}\,n_k - Q_{ij}\,n_i n_j\,Q^*_{kl}\,n_k n_l\right]$$

note: $\frac{dP_{dipole}}{d\Omega} \sim k^4\,\vec{p}^{\,2} \sim \frac{1}{\lambda^4}\cdot d^2 \sim \frac{1}{\lambda^2}\,\frac{d^2}{\lambda^2}$

$$\frac{dP_{quad}}{d\Omega} \sim k^6\,Q^2 \sim \frac{1}{\lambda^6}\cdot d^4 \sim \frac{1}{\lambda^2}\left(\frac{d^2}{\lambda^2}\right)^2$$

$\Rightarrow$ expansion is in d/λ, as advertised!

$\cancel{\rho = \frac{A}{e}\left(4\frac{x^2}{Q_0^2} + 4\frac{y^2}{Q_0^2} + \frac{z^2}{Q_0^2}\right) \quad Q_{xx} = \int dx\,dy\,dz\,(\ldots)}$

$\cancel{Q_i = Q_{ij}\,n_j =}$

$$\int d^3x\, J \;\propto\; \underset{\underset{\frac{1}{\lambda}}{\wr}}{\omega}\, \underset{\underset{ed}{\wr}}{\vec{p}} \;\propto e\,\frac{d}{\lambda}$$

$$\Rightarrow \vec{m} \;\propto\; \frac{1}{2}\int d^3x\, \vec{x}\times\vec{J} \sim d\int d^3x\, J \sim e\,\frac{d^2}{\lambda}$$

$$\Rightarrow \left.\frac{dP}{d\Omega}\right|_{\substack{\text{magnetic}\\ \text{dipole}}} \sim k^4 m^2 \sim \frac{1}{\lambda^4}\frac{d^4}{\lambda^2} = \frac{1}{\lambda^2}\left(\frac{d}{\lambda}\right)^4$$

Compare this with

$$\left.\frac{dP}{d\Omega}\right|_{EDM} \sim k^4 \underset{\underset{ed}{\wr}}{p^2} \sim \frac{1}{\lambda^4}\, d^2 \simeq \frac{1}{\lambda^2}\left(\frac{d}{\lambda}\right)^2$$

$$\Rightarrow \frac{\left(dP/d\Omega\right)_{\substack{\text{magnetic}\\ \text{dipole}}}}{\left(dP/d\Omega\right)_{EDM}} \sim \left(\frac{d}{\lambda}\right)^2 \ll 1. \Rightarrow \text{electric dipole radiates more}$$

This is due to $m \sim \frac{d^2}{\lambda}$, while $p \sim d$

$$\Rightarrow \frac{m}{p} \sim \frac{d}{\lambda} \ll 1.$$

$$\hat{n}\cdot\left[\left(\hat{n}\times(\hat{n}\times\vec{Q})\right)\times(\hat{n}\times\vec{Q}^{*})\right] = \hat{n}\cdot\left[\left(\hat{n}\,(\hat{n}\cdot\vec{Q})-\vec{Q}\right)\times\right.$$

$$\left.\times\left(\hat{n}\times\vec{Q}^{*}\right)\right] = \hat{n}\cdot\left[(\hat{n}\cdot\vec{Q})\left(\cancel{\hat{n}\,(\hat{n}\cdot\vec{Q}^{*})}-\vec{Q}^{*}\right)-\hat{n}\,|\vec{Q}|^{2}+\vec{Q}^{*}(\hat{n}\cdot\vec{Q})\right]$$

$$= \cancel{(\hat{n}\cdot\vec{Q})^{2}} - |\vec{Q}|^{2} + |\hat{n}\cdot\vec{Q}|^{2} = -\,Q_{ij}\,n_j\,Q^{*}_{ik}\,n_k\,+$$

$$+\quad Q_{ij}\,n_i\,n_j\;Q^{*}_{kl}\,n_k\,n_l$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{c\,k^{6}}{2(24\pi)^{2}\,\varepsilon_0}\left(Q_{ij}\,n_j\,Q^{*}_{ik}\,n_k - Q_{ij}\,n_i\,n_j\,Q^{*}_{kl}\,n_k\,n_l\right)}$$

One can integrate this using $Q_{ii}=0$ $\Rightarrow$

$$\boxed{P = \frac{c\,k^{6}}{1440\,\pi\,\varepsilon_0}\,|Q_{ij}|^{2}.}$$

Here $|Q_{ij}|^{2} \equiv Q_{ij}\,Q^{*}_{ij} = \sum_{i,j} |Q_{ij}|^{2}$.

<u>Example</u> : ellipsoidal oscillating charge distribution

$\Rightarrow Q_{zz} = Q_0, \qquad Q_{xx} = Q_{yy} = -Q_0/2$ as $Q_{ii}=0$

$Q_{ij}=0$ if $i\neq j$

$$\Rightarrow Q_{ij}\,n_j\,Q^{*}_{ik}\,n_k = +\left(\frac{Q_0}{2}\right)^{2}\left(n_x^{2}+n_y^{2}\right)+Q_0^{2}\,n_z^{2} =$$

$$= \frac{Q_0^{2}}{4}\sin^{2}\theta + Q_0^{2}\cdot\cos^{2}\theta$$

$$Q_{ij}\,n_i\,n_j = -\frac{Q_0}{2}\left(n_x^{2}+n_y^{2}\right)+Q_0\,n_z^{2} = -\frac{Q_0}{2}\sin^{2}\theta + Q_0\cos^{2}\theta$$

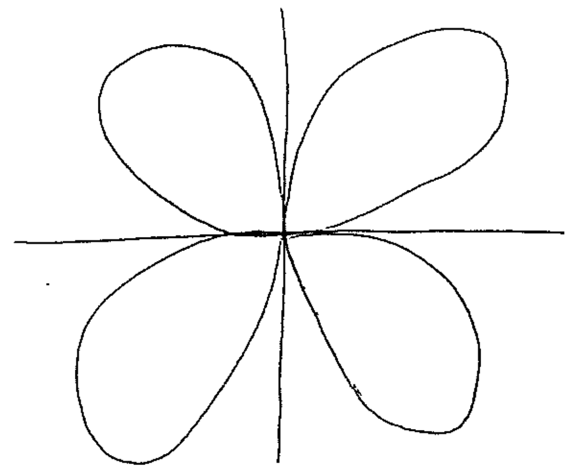
$$\Rightarrow \left|Q_{ij}\,n_i\,n_j\right|^{2} = \frac{Q_0^{2}}{4}\sin^{4}\theta + Q_0^{2}\cos^{4}\theta - Q_0^{2}\sin^{2}\theta\cos^{2}\theta$$

$$\Rightarrow Q_{ij} n_j Q_{ik} n_k - (Q_{ij} n_i n_j)^2 = \frac{Q_0^2}{4} \sin^2\theta \cos^2\theta +$$

$$+ Q_0^2 \sin^2\theta\cos^2\theta + Q_0^2 \sin^2\theta\cos^2\theta$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{c k^6 \cdot 9}{2(48\pi)^2 \varepsilon_0} Q_0^2 \sin^2\theta\cos^2\theta}$$

quadrupole radiation pattern:

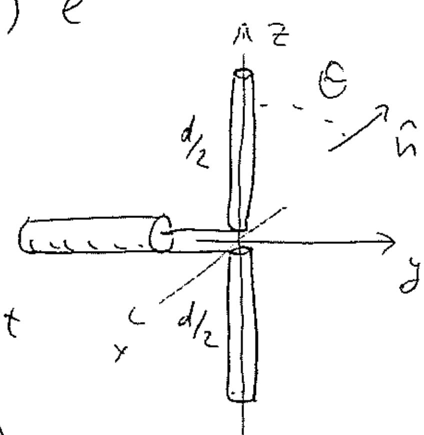


Center - Fed Linear Antenna.

In some cases we do not need to expand the vector-potential in the radiation zone:

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{J}(\vec{x}') \, e^{-ik\hat{n}\cdot\vec{x}'}$$

Consider a center-fed linear antena of length d:



$$\vec{J} = I \underbrace{\sin\left(\frac{kd}{2} - k|z|\right)}_{\theta\left(\frac{d}{2}-|z|\right)} \delta(x)\,\delta(y)\,\hat{z} \cdot e^{-i\omega t}$$

vanishes at the ends $(z = \pm d/2)$.

Plug it in:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot I\,\hat{z}\int_{-d/2}^{d/2} dz'\,\sin\left(\frac{kd}{2}-k|z'|\right)e^{-ikz'\cos\theta} =$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\,I\,\hat{z}\,\frac{1}{2i}\int_{-d/2}^{d/2} dz'\left(e^{i\left(\frac{kd}{2}-k|z'|\right)} - e^{-i\left(\frac{kd}{2}-k|z'|\right)}\right)e^{-ikz'\cos\theta} =$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\,I\,\hat{z}\,\frac{1}{2i}\left\{\frac{1}{-ik(1+\cos\theta)}\left(e^{-ik\frac{d}{2}\cos\theta} - e^{\frac{ikd}{2}}\right) - \right.$$

$$- \frac{1}{ik(1-\cos\theta)}\left(e^{-ik\frac{d}{2}\cos\theta} - e^{\frac{-ikd}{2}}\right) + \frac{1}{ik(1-\cos\theta)}\left(e^{i\frac{kd}{2}} - e^{ik\frac{d}{2}\cos\theta}\right)$$

$$\left. - \frac{1}{-ik(1+\cos\theta)}\left(e^{-i\frac{kd}{2}} - e^{ik\frac{d}{2}\cos\theta}\right)\right\} = \frac{\mu_0}{4\pi}\hat{z}\,I\,\frac{e^{ikr}}{kr}\cdot$$

$$\cdot\frac{1}{2i}\left\{\frac{1}{-i(1+\cos\theta)}\left[2\cos\left(\frac{kd}{2}\cos\theta\right) - 2\cos\left(\frac{kd}{2}\right)\right] + \right.$$

$$\left. + \frac{1}{i(1-\cos\theta)}\left[2\cos\left(\frac{kd}{2}\right) - 2\cos\left(\frac{kd}{2}\cos\theta\right)\right]\right\} = \frac{\mu_0}{2\pi}\hat{z}\,I\,\frac{e^{ikr}}{kr}\cdot$$

$$\cdot\frac{1}{\sin^2\theta}\left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right]$$ ~ have all powers of kd included.

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{ik}{\mu_0}\hat{n}\times\vec{A}\;;\quad \vec{E} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\vec{H}\times\hat{n}$$ (see dipole discussion)

acts on e^{ikr} only ~ radiation zone

$$\frac{dP}{d\Omega} = \underbrace{\frac{1}{2}\,\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right]}_{\vec{S}} \cdot \hat{n}\, r^2 = \frac{1}{2} r^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\, |\vec{H}|^2 = \frac{1}{2} r^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot$$

$$\cdot \frac{k^2}{\mu_0^2} \cdot \sin^2\theta\, |\vec{A}|^2 = \frac{1}{2} \cancel{r^2} \sqrt{\frac{\mu_0}{\varepsilon_0}}\, \frac{\cancel{k^2}}{\cancel{\mu_0^2}} \cdot \frac{\mu_0^2}{(2\pi)^2} I^2 \frac{1}{\cancel{k^2 r^2}} \frac{1}{\sin^4\theta} \cdot$$

$$\cdot \left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right]^2 \cdot \sin^2\theta$$

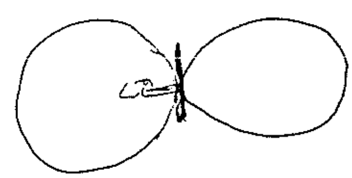
$$\Rightarrow \frac{dP}{d\Omega} = \frac{I^2}{8\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}} \left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right]^2$$

radiation of center-fed antenna!

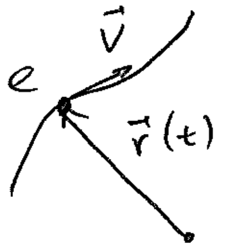
Multipole expansion in radiation zone was the expansion in $\frac{d}{\lambda} \sim kd \Rightarrow$ if $kd \ll 1$ the first term should give dipole contribution.

Expand for $kd \ll 1 \Rightarrow \dfrac{dP}{d\Omega} = \dfrac{I^2}{8\pi^2}\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\, \dfrac{k^4 d^4}{64} \cdot \sin^2\theta$

$\Rightarrow \dfrac{dP}{d\Omega} \propto k^4 d^4 \sin^2\theta$ ~ dipole radiation



Radiation by Moving Charges



Imagine a point charge (e) with an arbitrary (given) spatial trajectory $\vec{x} = \vec{r}(t)$.

What is the ratiation by this charge?

Working in Gaussian units we need to solve Maxwell equations (in Lorenz gauge):

$$\Box A^\mu = \frac{4\pi}{c} J^\mu .$$

The Green function satisfying

$$\Box_x G(x, x') = 4\pi c\, \delta^4(x - x')$$

was found before:
$$G_{ret}(x, x') = \frac{1}{|\vec{x} - \vec{x}'|}\, \delta\left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c}\right)$$

$\Rightarrow A^\mu(x) = \frac{1}{c^2}\int d^4x'\, G(x, x')\, J^\mu(x')$

$$\Rightarrow \quad A^\mu(x) = \frac{1}{c^2}\int d^4x' \frac{1}{|\vec{x} - \vec{x}'|}\, \delta\left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c}\right) J^\mu(x')$$

Let's plug in $J^\mu = (c\rho, \vec{J})$ with

$$\begin{cases} \rho(\vec{x}, t) = e\, \delta^3(\vec{x} - \vec{r}(t)) \\ \vec{J}(\vec{x}, t) = e\, \vec{v}(t)\, \delta^3(\vec{x} - \vec{r}(t)) \end{cases}$$

with $\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt}$

It is tempting to integrate over t' first, but remember that now $J^\mu_{(x')} \propto \delta^3(\vec{x}' - \vec{r}(t'))$!