Midterm Review

Separation of Variables in Cylindrical Coordinates

$$\nabla^2 \Phi(\rho, \varphi, z) = 0 \quad \Rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

A. z-independent case. $\quad \Phi(\rho, \varphi) = R(\rho)\,\Psi(\varphi)$

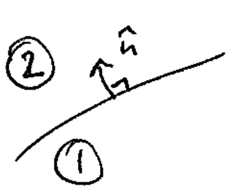
$$\begin{cases} R(\rho) = a\rho^{\nu} + b\rho^{-\nu} \\ \Psi(\varphi) = A\cos(\nu\varphi) + B\sin(\nu\varphi) \end{cases}, \quad \nu \neq 0$$

$$\begin{cases} R(\rho) = a_0 + b_0 \ln \rho \\ \Psi(\varphi) = A_0 + B_0 \varphi \end{cases}, \quad \nu = 0$$

for $\varphi \in [0, 2\pi] \Rightarrow \nu$ is integer $\Rightarrow \nu = n$

$$\Phi(\rho, \varphi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} \left[a_n \rho^{n} \sin(n\varphi + \alpha_n) + b_n \rho^{-n} \sin(n\varphi + \beta_n) \right].$$

boundary matching (from 1st semester):



$$\begin{cases} (\vec{E}_2 - \vec{E}_1)\cdot \hat{n} = \dfrac{\sigma}{\varepsilon_0} \\ E_{2t} = E_{1t} \end{cases}$$

B. z-dependent geometry

$$\Phi(\rho, \varphi, z) = R(\rho)\, Q(\varphi)\, Z(z)$$

$$\begin{cases} Z(z) = e^{\pm kz} \\ Q(\varphi) = e^{\pm i\nu\varphi} \\ R(\rho) = A\, J_\nu(k\rho) + B N_\nu(k\rho) \end{cases} \quad \text{OR} \quad \begin{cases} Z(z) = e^{\pm ikz} \\ Q(\varphi) = e^{\pm i\nu\varphi} \\ R(\rho) = C\, I_\nu(k\rho) + D\, K_\nu(k\rho) \end{cases}$$

Bessel functions of the 1st & 2nd kind

modified Bessel functions

pick one depending on convenient geometry

$x_{\nu n}$ is the n^{th} root of $J_\nu(z)$

$$\int_0^a d\rho \cdot \rho \cdot J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) J_\nu\left(x_{\nu m}\frac{\rho}{a}\right) = \frac{a^2}{2}\,\delta_{nm}\left[J_{\nu+1}(x_{\nu n})\right]^2$$

orthogonality relation

$J_\nu(z)$, $I_\nu(z)$ ~ finite at $z=0$

$I_\nu(z) \sim e^{z}$, $z \gg 1$

$K_\nu(z) \sim e^{-z}$, $z \gg 1$

$N_\nu(z)$, $K_\nu(z)$ ~ have singularities at $z=0$

J_ν, $N_\nu(z)$ ~ have roots

I_ν, K_ν ~ no non-trivial roots

Multipole Expansion

$\rho(\vec{x}')$

$$\boxed{\Phi(\vec{x}) = \frac{1}{\varepsilon_0} \sum_{l,m} \frac{1}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \varphi)}$$

with

$$\boxed{q_{lm} = \int d^3x' \, Y_{lm}^*(\theta', \varphi') \, r'^l \rho(\vec{x}')}$$

$$q_{00} = \frac{q_{net}}{\sqrt{4\pi}} ; \quad \vec{p} = \int d^3x \, \rho(\vec{x}) \, \vec{x} ; \quad Q_{ij} = \int d^3x \, \rho(\vec{x}) \left[3x_i x_j - r^2 \delta_{ij} \right].$$

dipole moment — quadrupole moment

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_{net}}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2} \sum_{ij} Q_{ij} \frac{x_i x_j}{r^5} + \ldots \right]$$

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_{net}\vec{x}}{r^3} + \frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{r^3} + \ldots \right], \quad \hat{n} = \frac{\vec{x}}{r}.$$

$$W = q \, \Phi_{ext}(0) - \vec{p}\cdot\vec{E}_{ext}(0) + \ldots$$ electrostatic energy in external field

Dielectrics: differential equations:

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho_{free} \\ \vec{\nabla}\times\vec{E} &= 0 \end{aligned}}$$

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

electric displacement

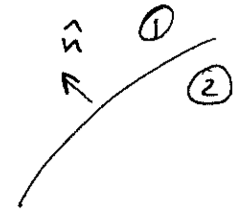
polarization (density of electric dipole moment)

LIH media: $\vec{D} = \varepsilon \vec{E} \Rightarrow$ if $\vec{E} = -\vec{\nabla}\Phi \Rightarrow \boxed{\nabla^2 \Phi = -\frac{\rho_{free}}{\varepsilon}}$

in general $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0} \vec{\nabla}\cdot(\vec{D} - \vec{P}) = \frac{\rho_{free} - \vec{\nabla}\cdot\vec{P}}{\varepsilon_0}$

$\Rightarrow$ $\rho_{bound} = -\vec{\nabla}\cdot\vec{P}$ ~ density of bound charges

Boundary conditions:



$$E_{1t} = E_{2t}$$

$$D_{1n} - D_{2n} = \sigma_{free}$$

$$P_{1n} - P_{2n} = -\sigma_{bound}$$

to solve boundary-value problems with dielectrics use $\nabla^2 \Phi = -\dfrac{\rho_{free}}{\varepsilon}$ and the techniques used before for solving Laplace / Poisson equations.

Electrostatic energy in dielectrics: (L/H)

$$W = \frac{1}{2}\int d^3x \; \vec{E}\cdot\vec{D}$$

or $$W = \frac{1}{2}\int d^3x \; \rho_{free}\cdot\Phi$$

Force

$$F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)_Q$$ (charges fixed on conductors)

$$F_\xi = +\left(\frac{\partial W}{\partial \xi}\right)_V$$ (potentials are fixed on conductors)

Magnetostatics

Maxwell equations give $\boxed{\begin{aligned}\vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{B} &= \mu_0 \vec{J}\end{aligned}}$

Continuity (charge conservation) reduces to $\boxed{\vec{\nabla}\cdot\vec{J} = 0}$

as $\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$ (Coulomb gauge, $\vec{\nabla}\cdot\vec{A} = 0$)

$$\Rightarrow \boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

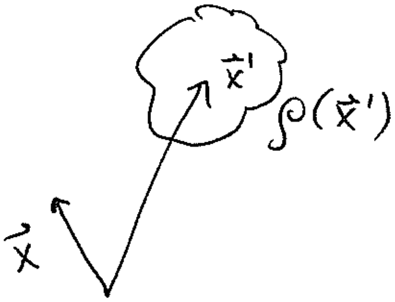
$$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}$$ Biot & Savart Law

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \Rightarrow \boxed{\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \int_S da \, \hat{n}\cdot\vec{J} = \mu_0 I}$$ Ampere's Law

Force: $\vec{F} = \int d^3x \, \vec{J}(\vec{x}) \times \vec{B}(\vec{x})$

Torque: $\vec{N} = \int d^3x \, \vec{x}\times\left[\vec{J}(\vec{x})\times\vec{B}(\vec{x})\right]$.

Magnetic multipole expansion:



$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}}$$

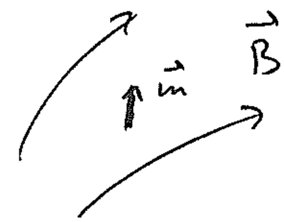
with the magnetic dipole moment

$$\boxed{\vec{m} = \frac{1}{2} \int d^3x \; \vec{x} \times \vec{J}(\vec{x})}.$$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}} \quad , \quad \hat{n} = \frac{\vec{x}}{|\vec{x}|}$$

Force on a magnetic dipole $\boxed{\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})}$

Potential energy: $\boxed{U = -\vec{m}\cdot\vec{B}}$



Magnetic materials

Def. Magnetic field $\vec{H}$:

$$\boxed{\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}}$$

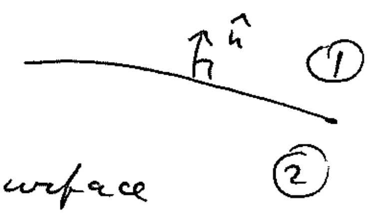
$\vec{M} = \frac{\text{magn. dip. moment}}{\text{volume}}$

LIH materials: $\boxed{\vec{B} = \mu \vec{H}}$

$\Rightarrow$ Maxwell equations are

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}} \qquad \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

Boundary matching:



$$\boxed{B_{1n} = B_{2n}} \qquad \boxed{\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}}$$

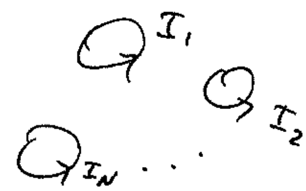
← surface (free) current density

Table: Boundary-value problems in magnetostatics

method	no Ferromagnetics $\vec{B} = \mu \vec{H}$	Ferromagnetics ($\vec{M} \neq 0$ for $\vec{H} = 0$)
$\vec{A}$ $\vec{B} = \vec{\nabla} \times \vec{A}$.	$\nabla^2 \vec{A} = -\mu \vec{J}$ $\Rightarrow \vec{A}(\vec{x}) = \frac{\mu}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{\lvert\vec{x} - \vec{x}'\rvert}$ (∞ space)	$\nabla^2 \vec{A} = -\mu_0 [\vec{J} + \vec{\nabla} \times \vec{M}]$ $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \frac{\vec{J}(\vec{x}') + \vec{\nabla}' \times \vec{M}(\vec{x}')}{\lvert\vec{x} - \vec{x}'\rvert}$ $+ \frac{\mu_0}{4\pi} \int_S da' \frac{\vec{K}(\vec{x}') + \vec{M}(\vec{x}') \times \hat{n}'}{\lvert\vec{x} - \vec{x}'\rvert}$ ← surface term
Φ_M (only for $\vec{J} = 0$) $\vec{H} = -\vec{\nabla} \Phi_M$	$\nabla^2 \Phi_M = 0$ Solve Laplace eq'n with b.c.'s.	$\nabla^2 \Phi_M = \vec{\nabla} \cdot \vec{M}$ $\Rightarrow \Phi_M(\vec{x}) = -\frac{1}{4\pi} \int_V d^3x' \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{\lvert\vec{x} - \vec{x}'\rvert}$ $+ \frac{1}{4\pi} \int_S da' \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{\lvert\vec{x} - \vec{x}'\rvert}$

Energy in magnetic field:

$$W = \frac{1}{2} \int d^3x \; \vec{H} \cdot \vec{B} = \frac{1}{2} \int d^3x \; \vec{J} \cdot \vec{A}$$



$$W = \frac{1}{2} \sum_i L_i I_i^2 + \frac{1}{2} \sum_{i \neq j} M_{ij} I_i I_j$$

↑ self-inductance ↑ mutual inductance