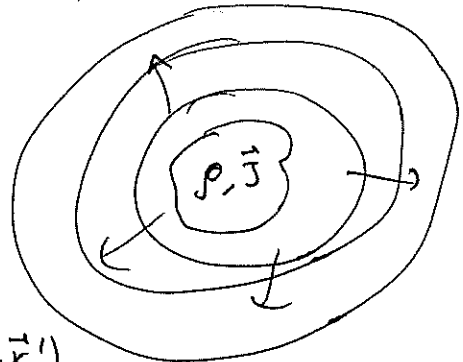
Last time

Magnetic Dipole and Electric Quadrupole (cont'd)

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\vec{J}(\vec{x}')\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}(\hat{n}\cdot\vec{x}')^n$$

$n=1$ term



$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int d^3x'\,\vec{J}(\vec{x}')(\hat{n}\cdot\vec{x}')$$

write $\vec{J}\,(\hat{n}\cdot\vec{x}') = \frac{1}{2}\left[(\hat{n}\cdot\vec{x}')\,\vec{J} + (\hat{n}\cdot\vec{J})\,\vec{x}'\right] + \frac{1}{2}(\vec{x}'\times\vec{J})\times\hat{n}$

(a) 2^{nd} term:

$$\boxed{\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\,\hat{n}\times\vec{m}}$$

$\vec{m}$ = magnetic dipole

$$\vec{H} = \frac{k^2}{4\pi}\frac{e^{ikr}}{r}(\hat{n}\times\vec{m})\times\hat{n}$$

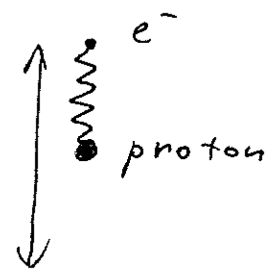
$$\vec{E} = \frac{k^2}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{e^{ikr}}{r}\,\vec{m}\times\hat{n}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,|\hat{n}\times\vec{m}|^2}$$

~ angular power distribution

Example Atom as a harmonic oscillator:
(hydrogen)

the position of electron is



$$\vec{x}(t) = \hat{z}\, r_0 \cos(\omega t) = \hat{z}\,\mathrm{Re}\left[r_0\, e^{-i\omega t}\right]$$

$$\Rightarrow \vec{p} = -e\,\vec{x}(t) = -e\,\hat{z}\, r_0\, \mathrm{Re}\, e^{-i\omega t}$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4\, |\vec{p}|^2 \sin^2\theta = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 \, .$$

$$\cdot\, e^2 r_0^2 \sin^2\theta$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{1}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{\omega^4}{c^2}\, e^2 r_0^2 \sin^2\theta$$

$$\Rightarrow P = \frac{1}{12\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{\omega^4}{c^2}\,|\vec{p}|^2 = \frac{1}{12\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{\omega^4}{c^2}\, e^2 r_0^2 = P$$

total radiated power.

~ Such classical atom would radiate energy

$\Rightarrow$ the electron would eventually fall on the proton $\Rightarrow$ need quantum mechanics to save the day!

Magnetic Dipole and Electric Quadrupole

Take the $n=1$ term:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-ik) \int d^3x' \, \vec{J}(\vec{x}') \, \hat{n}\cdot\vec{x}'$$

$$\vec{J}(\hat{n}\cdot\vec{x}') = \frac{1}{2}\left[(\hat{n}\cdot\vec{x}')\vec{J} + (\hat{n}\cdot\vec{J})\vec{x}'\right] + \frac{1}{2}(\vec{x}'\times\vec{J})\times\hat{n}$$

as $J_i x_j = \frac{1}{2}(J_i x_j + J_j x_i) + \frac{1}{2}(J_i x_j - J_j x_i)$

(a) take the 2nd term first: remember the magnetization

$\vec{M} = \frac{1}{2}\vec{x}\times\vec{J} \Rightarrow$ the 2nd term gives

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi} \frac{e^{ikr}}{r} \cdot \int d^3x' \, \hat{n}\times\vec{M}(\vec{x}') \Rightarrow$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi} \frac{e^{ikr}}{r} \cdot \hat{n}\times\vec{m}$$ ~ magnetic dipole radiation.

where $\vec{m}$ is the magnetic dipole moment:

$$\frac{dP}{d\Omega} = \frac{k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}|\hat{n}\times\vec{m}|^2$$

$$\vec{m} = \int d^3x' \, \vec{M}(\vec{x}')$$

$\Rightarrow$ can find $\vec{E}, \vec{H}$: $\vec{H} = \frac{k^2}{4\pi}\frac{e^{ikr}}{r}(\hat{n}\times\vec{m})\times\hat{n}$; $\vec{E} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} k^2 \vec{m}\times\hat{n}\frac{e^{ikr}}{r}$

(b) take the 1st term: $\frac{1}{2}\int d^3x' \left[n_i x_i' J_j + n_i J_i x_j'\right] =$

$$= \frac{1}{2}\int d^3x' \, \vec{J}\cdot\vec{\nabla}'\left(x_j'(\hat{n}\cdot\vec{x}')\right) = (\text{parts}) = -\frac{1}{2}\int d^3x' \, x_j'(\hat{n}\cdot\vec{x}') \underbrace{\vec{\nabla}'\cdot\vec{J}}_{i\omega\rho} = -\frac{i\omega}{2}\int d^3x' \, x_j'(\hat{n}\cdot\vec{x}')\rho(\vec{x}') \Rightarrow$$

$$\Rightarrow \vec{A}(\vec{x}) = -\frac{\mu_0}{4\pi}\,\frac{\omega k}{2}\,\frac{e^{ikr}}{r}\cdot\int d^3x'\cdot \vec{x}\,'\,(\hat{n}\cdot\vec{x}\,')\,\rho(\vec{x}\,')$$

electric quadrupole radiation

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} \approx \frac{ik}{\mu_0}\,\hat{n}\times\vec{A}\,; \quad \vec{E} = \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{H} =$$

$$\approx \frac{i}{\omega\varepsilon_0}\,ik\,\hat{n}\times\vec{H} = -\frac{1}{c\varepsilon_0}\,\frac{ik}{\mu_0}\,\hat{n}\times(\hat{n}\times\vec{A}) = -i\omega\,\hat{n}\times(\hat{n}\times\vec{A})$$

To find $\vec{H}$ need $\hat{n}\times\int d^3x'\,\vec{x}\,'\,(\hat{n}\cdot\vec{x}\,')\,\rho(\vec{x}\,')$.

Quadrupole moment tensor $Q_{ij} = \int d^3x\,(3x_ix_j - r^2\delta_{ij})\,\rho$

$$\Rightarrow \left[\hat{n}\times\int d^3x'\,\vec{x}\,'\,(\hat{n}\cdot\vec{x}\,')\,\rho(\vec{x}\,')\right]_i = \varepsilon_{ijk}\,n_j\int d^3x'\cdot x'_k\,.$$

$$\cdot\, n_l\cdot x'_l\,\rho(\vec{x}\,') = \frac{1}{3}\,\varepsilon_{ijk}\,n_j\,n_l\,Q_{kl} = \frac{1}{3}\,\hat{n}\times\vec{Q}$$

with $(\vec{Q})_i = Q_{ij}\,n_j$ $\Rightarrow$ $\vec{H} = -\frac{i}{8\pi}\,\omega k^2\,\frac{e^{ikr}}{r}\cdot\frac{1}{3}\,\hat{n}\times\vec{Q}$

$$\vec{E} = -\frac{1}{c\varepsilon_0}\,\hat{n}\times\vec{H} = \frac{i}{24\pi}\,\frac{\omega k^2}{c\varepsilon_0}\,\frac{e^{ikr}}{r}\,\hat{n}\times(\hat{n}\times\vec{Q})$$

Radiated power is (time averaged)

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{n}\cdot(\vec{E}\times\vec{H}^*)\right] = -\frac{1}{2}\,\cancel{r^2}\,\frac{1}{3^2(8\pi)^2}\,\frac{\omega^2k^4}{c\varepsilon_0}\,\frac{1}{\cancel{r^2}}\,.$$

$$\cdot\,\hat{n}\cdot\left((\hat{n}\times(\hat{n}\times\vec{Q}))\times(\hat{n}\times\vec{Q}^*)\right)$$

$$\vec{E} = -\frac{1}{c\varepsilon_0}\hat{n}\times\vec{H} \Rightarrow \frac{dP}{d\Omega} = \frac{r^2}{2}\mathrm{Re}\left[\hat{n}\cdot(\vec{E}\times\vec{H}^*)\right] = \frac{r^2}{2}\left(\frac{-1}{c\varepsilon_0}\right)$$

$$\cdot\, \mathrm{Re}\left[\hat{n}\cdot\left((\hat{n}\times\vec{H})\times\vec{H}^*\right)\right] = \frac{r^2}{2c\varepsilon_0}|\vec{H}|^2 = \frac{\cancel{r^2}}{2c\varepsilon_0}\cdot\frac{1}{64\pi^2}\,\omega^2 k^4 \frac{1}{\cancel{r^2}}\frac{1}{9}\,\cdot$$

$$-\vec{H}^*\times(\hat{n}\times\vec{H}) = -\hat{n}|\vec{H}|^2 + \vec{H}\underbrace{(\hat{n}\cdot\vec{H}^*)}_{=0} = -\hat{n}|\vec{H}|^2$$

$$\cdot\,|\hat{n}\times\vec{Q}|^2 = \frac{\omega^2 k^4}{18\,c\varepsilon_0\, 64\pi^2}\,|\hat{n}\times\vec{Q}|^2$$

$$\delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}$$

$$(\hat{n}\times\vec{Q})^2 = |\varepsilon_{ijk}\, n_j\, Q_{kl}\, n_l|^2 = \varepsilon_{ijk}\, n_j\, Q_{kl}\, n_l \cdot \varepsilon_{ij'k'}\cdot$$

$$\cdot\, n_{j'}\, Q^*_{k'l'}\, n_{l'} = n_j Q_{kl} n_l\, n_j Q^*_{kl'} n_{l'} - n_j Q_{kl} n_l\, n_k Q^*_{jl'} n_{l'}$$

$$= Q_{kl} n_l Q^*_{kl'} n_{l'} - Q_{kl} n_k n_l\, Q^*_{ij} n_i n_j$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{ck^6}{2(24\pi)^2\varepsilon_0}\left[Q_{ij}n_j\, Q^*_{ik}\, n_k - Q_{ij}\, n_i n_j\, Q^*_{kl}\, n_k\, n_l\right]$$

note: $\frac{dP_{dipole}}{d\Omega} \sim k^4 \vec{p}^{\,2} \sim \frac{1}{\lambda^4}\cdot d^2 \sim \frac{1}{\lambda^2}\frac{d^2}{\lambda^2}$

$$\frac{dP_{quad}}{d\Omega} \sim k^6\, Q^2 \sim \frac{1}{\lambda^6}\cdot d^4 \sim \frac{1}{\lambda^2}\left(\frac{d^2}{\lambda^2}\right)^2$$

a expansion is in d/λ, as advertised!

~~$\rho\left(4\frac{x^2}{Q_0^2}+4\frac{y^2}{Q_0^2}+\frac{z^2}{Q_0^2}\right)$ $Q_{xx} = \int dx\,dy\,dz\,(3x^2 - r^2)\ \frac{}{2}\left(\frac{x^2+y^2}{Q_0^2/4}+\frac{z^2}{Q_0^2}\right)$~~

~~$Q_i = Q_{ij}n_j =$~~

Alternatively,

$$\hat{n}\cdot\left[\left(\hat{n}\times(\hat{n}\times\vec{Q})\right)\times(\hat{n}\times\vec{Q}^*)\right] = \hat{n}\cdot\left[\left(\hat{n}(\hat{n}\cdot\vec{Q})-\vec{Q}\right)\times\right.$$

$$\left.\times(\hat{n}\times\vec{Q}^*)\right] = \hat{n}\cdot\left[(\hat{n}\cdot\vec{Q})\left(\cancel{\hat{n}(\hat{n}\cdot\vec{Q}^*)-\vec{Q}^*}\right)^0 - \hat{n}|\vec{Q}|^2 + \vec{Q}^*(\hat{n}\cdot\vec{Q})\right]$$

$$= \cancel{(\hat{n}\cdot\vec{Q})^2} - |\vec{Q}|^2 + (\hat{n}\cdot\vec{Q})^2 = -Q_{ij}n_j Q^*_{ik}n_k +$$

$$+ Q_{ij}n_i n_j Q^*_{kl} n_k n_l$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{ck^6}{2(24\pi)^2\varepsilon_0}\left(Q_{ij}n_j Q^*_{ik}n_k - Q_{ij}n_i n_j Q^*_{kl}n_k n_l\right)}$$

One can integrate this using $Q_{ii}=0 \Rightarrow$

$$\boxed{P = \frac{ck^6}{1440\pi\varepsilon_0}|Q_{ij}|^2.}$$

Here $|Q_{ij}|^2 \equiv Q_{ij}Q^*_{ij} = \sum_{ij}|Q_{ij}|^2$.

Example: ellipsoidal oscillating charge distribution

$$\Rightarrow Q_{zz} = Q_0, \qquad Q_{xx} = Q_{yy} = -Q_0/2 \quad \text{as } Q_{ii}=0$$

$$Q_{ij}=0 \text{ if } i\neq j$$

$$\Rightarrow Q_{ij}n_j Q_{ik}n_k = +\left(\frac{Q_0}{2}\right)^2(n_x^2+n_y^2) + Q_0^2 n_z^2 =$$

$$= \frac{Q_0^2}{4}\sin^2\theta + Q_0^2\cos^2\theta$$

$$Q_{ij}n_i n_j = -\frac{Q_0}{2}(n_x^2+n_y^2) + Q_0 n_z^2 = -\frac{Q_0}{2}\sin^2\theta + Q_0\cos^2\theta$$

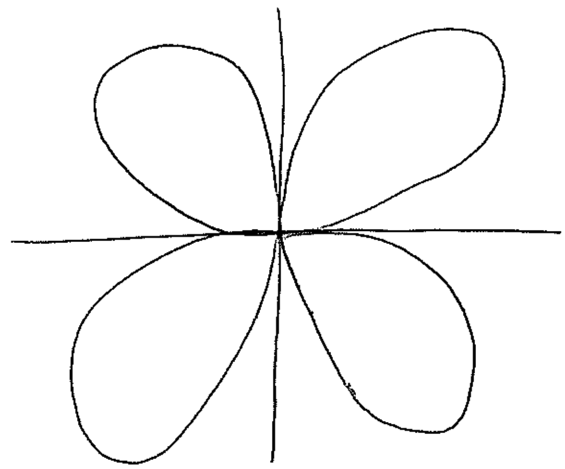
$$\Rightarrow |Q_{ij}n_i n_j|^2 = \frac{Q_0^2}{4}\sin^4\theta + Q_0^2\cos^4\theta - Q_0^2\sin^2\theta\cos^2\theta$$

$$\Rightarrow Q_{ij} n_j Q_{ik} n_k - \left(Q_{ij} n_i n_j \right)^2 = \frac{Q_0^2}{4} \sin^2\theta \cos^2\theta +$$

$$+ Q_0^2 \sin^2\theta \cos^2\theta + Q_0^2 \sin^2\theta \cos^2\theta$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{c k^6 \cdot 9}{2(48\pi)^2 \varepsilon_0} Q_0^2 \sin^2\theta \cos^2\theta}$$

quadrupole radiation pattern:

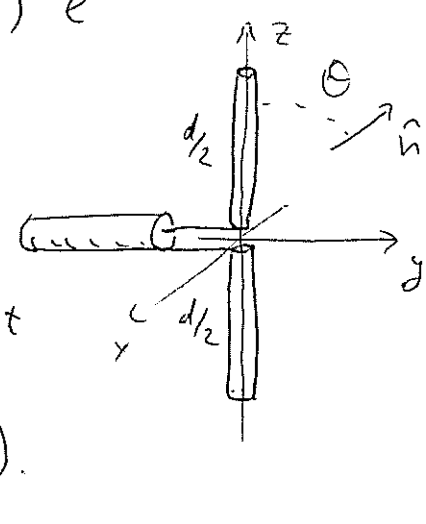


Center-Fed Linear Antenna

In some cases we do not need to expand the vector-potential in the radiation zone:

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{J}(\vec{x}') \, e^{-ik\hat{n}\cdot\vec{x}'}$$

Consider a center-fed linear antena of length d:



$$\vec{J} = I \sin\left(\frac{kd}{2} - k|z|\right) \delta(x)\,\delta(y)\,\hat{z} \cdot e^{-i\omega t}$$

vanishes at the ends $(z = \pm d/2)$.