Lorentz Group

Work in Minkowski space, $\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

$\eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu{}^\rho$; $\quad x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{x})$, $x^\mu = (t, \vec{x})$.

Def. Set of linear (real) transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

forms the Lorentz group if

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$$

(proper time is preserved). $\quad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ metric tensor

Example $\quad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for boosts along x^1-axis.

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$$

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta = \eta_{\alpha\beta} x^\alpha x^\beta$$

$$\Rightarrow \quad \boxed{\eta_{\alpha\beta} = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta}$$

or, equivalently, $\quad \eta = \Lambda^T \eta \Lambda$

as $\eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu{}^\rho \Rightarrow \eta \cdot \eta = \mathbb{1}$

$$\Rightarrow \quad \eta \cdot \eta = \mathbb{1} = \eta \Lambda^T \eta \Lambda \quad \Rightarrow \quad \boxed{\Lambda^{-1} = \eta \Lambda^T \eta}$$

$$\Rightarrow \quad \eta = \Lambda \eta \Lambda^T$$ (multiply by $\Lambda \Rightarrow \mathbb{1} = \Lambda \eta \Lambda^T \eta$) & by η on the right.

Why is this set a group? Λ_1, Λ_2 ~ different L. tr.

(i) $\Lambda = \Lambda_2 \cdot \Lambda_1 \quad \Rightarrow \quad \Lambda^\mu{}_\nu = \Lambda_2{}^\mu{}_\rho \Lambda_1{}^\rho{}_\nu$

$$\eta = \Lambda_1^T \eta \Lambda_1 \,, \quad \eta = \Lambda_2^T \eta \Lambda_2$$

$$\Rightarrow \quad (\Lambda_2 \Lambda_1)^T \eta (\Lambda_2 \Lambda_1) = \Lambda_1^T \underbrace{\Lambda_2^T \eta \Lambda_2}_{\eta} \Lambda_1 = \Lambda_1^T \eta \Lambda_1 = \eta$$

$\Rightarrow \quad \Lambda \in$ Lorentz group

(ii) $\Lambda_1 \cdot (\Lambda_2 \cdot \Lambda_3) = (\Lambda_1 \cdot \Lambda_2) \cdot \Lambda_3$ trivially true for matrices

(iii) Identity: $\delta^\mu{}_\nu = \mathbb{1}$ exists.

(iv) $\forall \Lambda \in$ Lorentz group there exists

$$\Lambda^{-1} = \eta \Lambda^T \eta \; : \; \Lambda \Lambda^{-1} = \Lambda^{-1} \Lambda = \mathbb{1}.$$

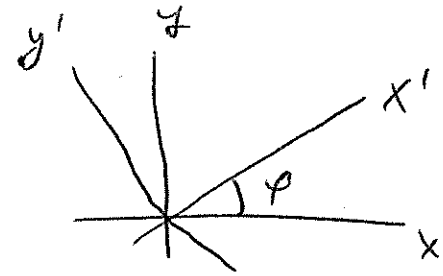
$\Rightarrow$ Lorentz group is a group.

Examples of Lorentz group elements:

(1) Usual Lorentz transformation:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(2) Rotation in $x-y$ plane:



$$x \to x' = x\cos\varphi + y\sin\varphi$$

$$y \to y' = -x\sin\varphi + y\cos\varphi$$

$$= \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad \Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(3) Parity: $\vec{x} \to -\vec{x}$, $\mathbb{P}$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

(4) Time reversal, $\mathbb{T}$: $t \to -t$,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}.$$

Now, $\eta = \Lambda^T \eta \Lambda \Rightarrow \cancel{\det} \eta = \underbrace{\det \Lambda^T}_{= \det \Lambda} \cancel{\det \eta} \det \Lambda$

$\Rightarrow$ $\det \Lambda = \pm 1$. "+" proper, "−" improper LT's.

Also, $1 = \eta_{00} = \Lambda^\mu{}_0 \Lambda^\nu{}_0 \eta_{\mu\nu} = \Lambda^0{}_0 \Lambda^0{}_0 - \Lambda^i{}_0 \Lambda^i{}_0$

$\Rightarrow 1 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 \Rightarrow |\Lambda^0{}_0| \geq 1 \Rightarrow$

$\Rightarrow$ either $\boxed{\Lambda^0{}_0 \geq 1}$ or $\boxed{\Lambda^0{}_0 \leq -1}$.

orthochronous   non-orthocronous

4 types of transformations:

$\det \Lambda = +1$, $\Lambda^0{}_0 \geq 1$ (e.g. boosts, rotations)

$\det \Lambda = +1$, $\Lambda^0{}_0 \leq -1$ (e.g. full inversion $x^\mu \to -x^\mu$)

$\det \Lambda = -1$, $\Lambda^0{}_0 \geq 1$ (parity $\mathbb{P}$)

$\det \Lambda = -1$, $\Lambda^0{}_0 \leq -1$ (time reversal $\mathbb{T}$).

Now; $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \Rightarrow x' = \Lambda \cdot x \Rightarrow x = \Lambda^{-1} \cdot x'$

$\Rightarrow x^{\nu} = (\Lambda^{-1})^{\nu}{}_{\mu} x'^{\mu} \Rightarrow \dfrac{\partial x^{\nu}}{\partial x'^{\mu}} = (\Lambda^{-1})^{\nu}{}_{\mu}$.

as $\eta^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \eta^{\alpha\beta} \Rightarrow \delta^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda_{\nu\beta} \eta^{\alpha\beta}$

$= \Lambda^{\mu}{}_{\alpha} \Lambda_{\nu}{}^{\alpha} = \Lambda^{\mu}{}_{\alpha} \cdot (\Lambda^{-1})^{\alpha}{}_{\nu} \Rightarrow$

$$(\Lambda^{-1})^{\alpha}{}_{\nu} = \Lambda_{\nu}{}^{\alpha}$$

Thus $\dfrac{\partial x^{\nu}}{\partial x'^{\mu}} = (\Lambda^{-1})^{\nu}{}_{\mu} = \Lambda_{\mu}{}^{\nu}$