Final Review

Electrostatics

Maxwell equations reduce to

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0} \qquad \boxed{\vec{\nabla}\times\vec{E} = 0}$$

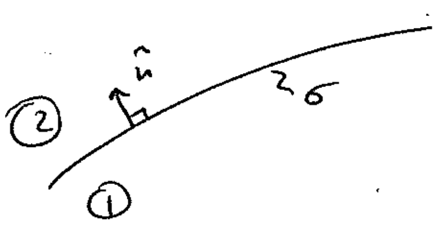
$$\Rightarrow \quad \vec{E} = -\vec{\nabla}\Phi \quad \Rightarrow \quad \boxed{\nabla^2\Phi = -\rho/\varepsilon_0}$$ Poisson eq'n

if $\rho = 0 \Rightarrow \boxed{\nabla^2\Phi = 0}$ Laplace eq'n

Gauss's Law in integral form:

$$\boxed{\int_S da\ \hat{n}\cdot\vec{E} = Q/\varepsilon_0} \quad \Rightarrow$$

boundaries:



$$\boxed{\hat{n}\cdot(\vec{E}_2 - \vec{E}_1) = \sigma/\varepsilon_0}$$

$$\boxed{E_{2t} = E_{1t}}$$

Solution of Poisson eq'n in infinite space:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\ \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}$$

Note that

$$\nabla^2 \frac{1}{|\vec{x}-\vec{x}\,'|} = -4\pi\,\delta^3(\vec{x}-\vec{x}\,')$$

Solving Poisson equation: use "Master formula":

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x'\, G(\vec{x},\vec{x}')\,\rho(\vec{x}') + \frac{1}{4\pi}\int_{S'} da'\left[G(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial G}{\partial n'}\right]}$$

where $\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta^3(\vec{x}-\vec{x}')$

↖ Green function

Dirichlet b.c. problem: Φ given on S'

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x'\, G_D(\vec{x},\vec{x}')\,\rho(\vec{x}') - \frac{1}{4\pi}\int_{S'} da'\,\Phi(\vec{x}')\frac{\partial G}{\partial n'}}$$

G_D ~ Dirichlet Green ftn (vanishes on S')

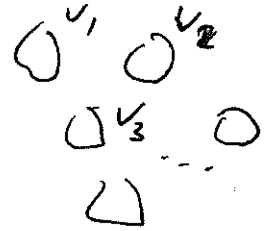
Neumann b.c. problem: $\frac{\partial\Phi}{\partial n}$ given on S'

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x'\, G_N(\vec{x},\vec{x}')\,\rho(\vec{x}') + \frac{1}{4\pi}\int_{S'} da'\, G_N(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'} + \langle\Phi\rangle_{surf.}}$$

G_N ~ Neumann Green ftn, $\left.\frac{\partial G_N}{\partial n}\right|_{\text{on } S} = -\frac{4\pi}{S}$.

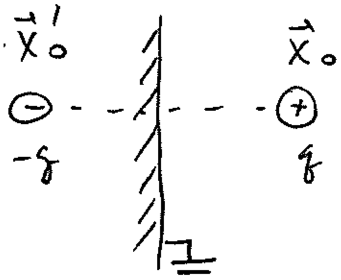
Electrostatic energy: $\mathcal{E} = \int d^3x \, \frac{\varepsilon_0}{2} \vec{E}^{\,2} = \frac{1}{2}\int d^3x \, \Phi(\vec{x}) \rho(\vec{x})$

Capacitance: $Q_i = \sum_j c_{ij} V_j$

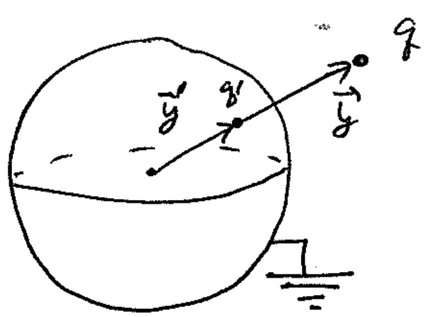


c_{ii} ~ capacitances

Method of Images



$\Rightarrow \quad \Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{x}_0|} - \frac{1}{|\vec{x}-\vec{x}_0'|}\right]$



$q' = -q\frac{R}{y}$, $y' = R^2/y$

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{y}|} - \frac{R}{y}\frac{1}{|\vec{x}-\frac{R^2}{y^2}\vec{y}|}\right].$$

$\Rightarrow$ can be used to find Dirichlet Green function

$\Rightarrow$ outside the sphere we have:

$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{R}{x'}\frac{1}{|\vec{x}-\frac{R^2}{x'^2}\vec{x}'|}.$$

ditto inside the sphere, for a half-space, etc.

Separation of Variables and Special Functions

$\{u_n(x)\}$ complete orthonormal set:

$$\boxed{\int_a^b dx\, u_n^*(x)\, u_m(x) = \delta_{mn}}$$ orthonormality

$$\boxed{\sum_n u_n^*(x)\, u_n(y) = \delta(x-y)}$$ completeness

$\Rightarrow$ $\forall f(x)$ can be expanded $f(x) = \sum_n a_n u_n(x)$

if $\hat{L}\, u_n(x) = \lambda_n u_n(x)$

$$\Rightarrow \boxed{G(x, x') = -4\pi \sum_n \frac{u_n(x)\, u_n^*(x')}{\lambda_n}}$$

e.g. $u_n(x) = \frac{1}{\sqrt{a}}\, e^{i\frac{2\pi n x}{a}}$ $\Rightarrow$ $\frac{1}{a}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{a}(x-x')} = \delta(x-x')$

Fourier integral:

$$\boxed{\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, e^{ikx} A(k) \\ A(k) &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-ik\cdot x} f(x) \end{aligned}}$$

a useful relation:

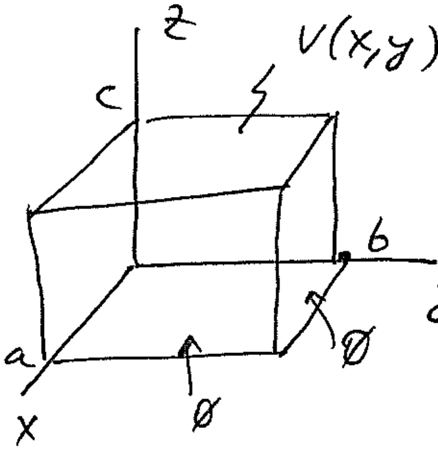
$$\boxed{\int_{-\infty}^{\infty} dk\, e^{ik\cdot(x-x')} = 2\pi\, \delta(x-x')}$$

Solving Laplace equation in rectangular coordinates

$$\Phi(\vec{x}) = X(x)\,Y(y)\,Z(z)$$

$$\Rightarrow \begin{cases} X(x) = e^{\pm i\alpha x} \\ Y(y) = e^{\pm i\beta y} \\ Z(z) = e^{\pm \gamma z} \end{cases}, \qquad \gamma^2 = \alpha^2 + \beta^2$$

Example



$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi m y}{b}\right) \cdot \sinh\left(z\,\gamma_{nm}\right),$$

$$\gamma_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}.$$

$$A_{nm} = \frac{4}{ab\,\sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy\; V(x,y) \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi m y}{b}\right).$$

Solving Laplace equation in spherical coordinates

$$\Phi(r,\theta,\varphi) = \frac{u(r)}{r} P(\theta)\, Q(\varphi)$$

$$\Rightarrow Q(\varphi) = e^{\pm i m\varphi}, \qquad \frac{u(r)}{r} = A_{em} r^{e} + B_{em} r^{-e-1}.$$

(A) Azimuthally symmetric case $(m=0)$

$$\Phi(\vec{x}) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-\ell-1}\right] P_\ell(\cos\theta)$$

$P_\ell(\cos\theta)$ ~ Legendre polynomials

(B) Azimuthally asymmetric case $(m \neq 0)$

$$\Phi(\vec{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell m} r^\ell + B_{\ell m} r^{-\ell-1}\right] Y_{\ell m}(\theta,\varphi)$$

$Y_{\ell m}(\theta,\varphi)$ ~ spherical harmonics

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \, Y^*_{\ell' m'}(\theta,\varphi) \, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi') \, Y_{\ell m}(\theta,\varphi) = \delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta')$$

$$Y_{\ell 0}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}} \, P_\ell(\cos\theta)$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell m}(\theta',\varphi') \, Y_{\ell m}(\theta,\varphi).$$

Green function of Poisson eq'n in unlimited space expanded over $Y_{\ell m}$'s.