Last time

(Def.) Interval: $s_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2$.

Infinitesimal interval: $ds^2 = c^2 dt^2 - d\vec{x}^2$.

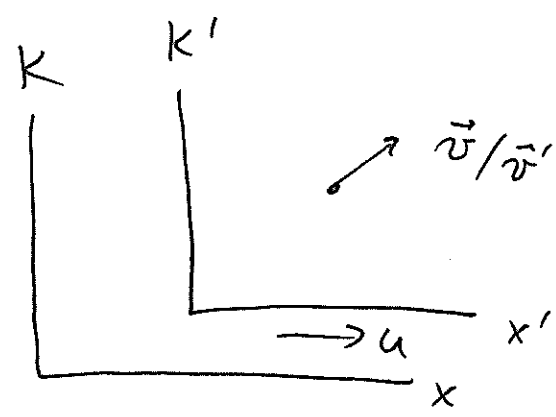
(Def.) Proper time: $\boxed{d\tau = \frac{ds}{c}}$

$$\Rightarrow \quad \tau_2 - \tau_1 = \int_{t_1}^{t_2} dt \sqrt{1-\beta^2(t)} \quad \Rightarrow \Delta\tau \leq \Delta t$$

Lorentz contraction: $\boxed{l = l_0 \sqrt{1 - \frac{v^2}{c^2}}}$

proper length
(length in rest frame)

Velocity Transformations:



$$\boxed{\begin{aligned} v_x &= \frac{v_x' + u}{1 + \frac{u v_x'}{c^2}} \\ v_{y,z} &= v'_{y,z} \frac{\sqrt{1-\frac{u^2}{c^2}}}{1 + \frac{u v_x'}{c^2}} \end{aligned}} \quad \Rightarrow$$

angle transformation

$$\tan\theta = \frac{v' \sin\theta'}{\gamma(v'\cos\theta' + u)}.$$

Four - vectors.

We have seen one example: $x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \Leftrightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \Rightarrow$$

$$\Rightarrow \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Lambda^{\mu}{}_{\nu}.$$

A general Lorentz transformation can be

- a boost along x, y, or z axis ($\forall$ velocity β)
- a rotation around the x, y, or z axis ($\forall$ angle θ)

Definition A 4-vector A^μ is a set of 4 quantities (A^0, A^1, A^2, A^3), which under Lorentz transformations transform as $\boxed{A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu}$ (just like x^μ)

$A'^\mu = \Lambda^\mu{}_\nu A^\nu$

$\mu, \nu = 0, 1, 2, 3$

summation over ν is implied.

Example boost along the x-axis:

$$\begin{pmatrix} A^{0'} \\ A^{1'} \\ A^{2'} \\ A^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}.$$

$\Rightarrow$ A^μ, $\mu = 0, 1, 2, 3$ is a contravariant vector if it transforms according to:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu \quad \text{(equivalent to above)}$$

$\Rightarrow$ B_μ, $\mu = 0, \dots, 3$ is a covariant vector if

$$B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu$$

Example: $\frac{\partial \varphi}{\partial x^\mu}$ is a covariant vector as

$$\frac{\partial \varphi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \varphi}{\partial x^\nu}.$$

One can define tensors by

$$A'^\mu B'^\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} \cdot A^\alpha B^\beta$$

$\Rightarrow$ rank two contravariant tensor would be $C'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} C^{\alpha\beta}$, etc.

In general $T^{\nu_1 \dots \nu_m}_{\mu_1 \dots \mu_n}$ transforms as $A_{\mu_1} \dots A_{\mu_n} B^{\nu_1} \dots B^{\nu_m}$.

Definition Scalar (inner) product of 2 vectors is defined by $A_\mu \cdot B^\mu$ (summation assumed)

Let's prove that it's Lorentz-invariant:

$$A'_\mu \cdot B'^\mu = \frac{\partial x^\alpha}{\partial x'^\mu} A_\alpha \frac{\partial x'^\mu}{\partial x^\beta} B^\beta = \frac{\partial x^\alpha}{\partial x^\beta} A_\alpha B^\beta = \delta^\alpha_\beta \cdot A_\alpha B^\beta = A_\alpha B^\alpha . \quad \text{Q.E.D.}$$

The interval is a scalar: (it's Lorentz-invariant)

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

Define the <u>metric tensor</u> by

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

(Minkowski space) Cartesian coordinates

Note that $dx_\mu dx^\mu$ is also a Lorentz-scalar.

Identifying $dx_\mu = g_{\mu\nu} dx^\nu$ we see that

$g_{\mu\nu}$ lowers indices of 4-vectors, tensors, etc.

Example $x^\mu = (ct, \vec{x})$, $x_\mu = g_{\mu\nu} x^\nu = (ct, -\vec{x})$.

contravariant, covariant

(Def.) $g^{\mu\nu}$ is an inverse matrix to $g_{\mu\nu}$: $\boxed{g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}{}_{\nu}}$

$$A_\mu = g_{\mu\nu} A^\nu \Rightarrow A^\mu = g^{\mu\nu} A_\nu$$

invert $g_{\mu\nu}$ matrix

where $\delta^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(e.g. $x_\mu = g_{\mu\nu} x^\nu, \ldots$)

$\Rightarrow g^{\mu\nu}$ raises indices of 4-vetors, tensors, etc.

$$\delta^{\mu}{}_{\nu} = g^{\mu\alpha} \cdot g_{\alpha\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}$$

(only for Minkowski space!)

indeed, as $A_\mu B^\mu = \delta^{\mu}{}_{\nu} \cdot A_\mu B^\nu = g^{\mu}{}_{\nu} A_\mu B^\nu$.

(Define) an abbreviated notation: $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$$

$\Rightarrow \partial_\mu \varphi$ is a covariant vector

($\partial^\mu \varphi$ is a contravariant vector (check!))

$$\left(\frac{\partial \varphi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \varphi}{\partial x^\nu} \right)$$

$\partial_\mu A^\mu$ is Lorentz-invariant

4d Laplace operator $\dfrac{\partial^2}{c^2 \partial t^2} - \vec{\nabla}^2 = \underbrace{\partial_\mu \partial^\mu}_{= \frac{\partial}{\partial x^0}\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x^2}\frac{\partial}{\partial x_2}}$ is

(d'Alembertian)

also Lorentz-invariant. (can you prove this?)

4-velocity

Let's define a 4-vector for velocity:

$$dx^\mu = (dx^0, dx^1, dx^2, dx^3) \Rightarrow v^\mu \stackrel{?}{=} \frac{dx^\mu}{dt} \; ?$$