Last time

Four-vectors

Def. A^μ is a <u>contravariant</u> 4-vector if it transforms as

$$\boxed{A^\mu \to A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu}$$

under Lorentz transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

boosts & rotations

B_μ is a <u>covariant</u> 4-vector if

$$\boxed{B_\mu \to B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu}$$

$A^\mu B_\mu = A'^\mu B'_\mu$ is a Lorentz-scalar

$A^\mu B^\nu \sim$ rank-2 tensor

$A^\mu B^\nu C^\rho \sim$ rank-3 tensor, etc.

Def. $ds^2 \equiv g_{\mu\nu}\, dx^\mu dx^\nu \Rightarrow g_{\mu\nu}$ is the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

in Minkowski space-time, using Cartesian coordinates

$\Rightarrow$ $A_\mu = g_{\mu\nu} A^\nu$ $\Rightarrow g_{\mu\nu}$ lowers indices

Defining $\delta^\mu_\nu = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases}$ (Kronecker delta in 4 dim)

we defined $g^{\mu\nu}$ by

$$\boxed{g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu} \Rightarrow g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$\Rightarrow \boxed{A^\mu = g^{\mu\nu} A_\nu} \Rightarrow g^{\mu\nu}$ raises indices

Examples: $x^\mu = (ct, \vec{x})$ ~ contravariant 4-vector

$x_\mu = g_{\mu\nu} x^\nu = (ct, -\vec{x})$ ~ covariant 4-vector.

Def. $\boxed{\partial_\mu \equiv \frac{\partial}{\partial x^\mu}}$, $\boxed{\partial^\mu \equiv \frac{\partial}{\partial x_\mu}}$.

$\Rightarrow \partial_\mu \varphi(x)$ is a covariant 4-vector ($\varphi(x)$ is a scalar)

$\partial_\mu A^\mu$ is a Lorentz scalar ($A^\mu(x)$ ~ 4-vector field)

$$\partial_\mu A^\mu = \frac{1}{c}\frac{\partial A^0}{\partial t} + \underbrace{\partial_i}_{=\vec{\nabla}^i} A^i = \frac{1}{c}\frac{\partial A^0}{\partial t} + \vec{\nabla}\cdot\vec{A}$$

$\partial_\mu \partial^\mu$ = Lorentz-invariant $\Rightarrow \partial_\mu \partial^\mu = \partial_0 \partial^0 + \partial_i \partial^i =$

$= (\partial^0)^2 - (\vec{\nabla})^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \equiv \Box$ ~ d'Alambertian operator (wave equation operator)

Definition Scalar (inner) product of 2 vectors is defined by $A_\mu \cdot B^\mu$ (summation assumed)

Let's prove that it's Lorentz-invariant:

$$A'_\mu \cdot B'^\mu = \frac{\partial x^\alpha}{\partial x'^\mu} A_\alpha \frac{\partial x'^\mu}{\partial x^\beta} B^\beta = \frac{\partial x^\alpha}{\partial x^\beta} A_\alpha B^\beta = \delta^\alpha_\beta \cdot$$

$$\cdot A_\alpha B^\beta = A_\alpha B^\alpha . \quad \text{Q.E.D.}$$

The interval is a scalar: (it's Lorentz-invariant)

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

Define the <u>metric tensor</u> by

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(Minkowski space) Cartesian coordinates

Note that $dx_\mu dx^\mu$ is also a Lorentz-scalar.

Identifying $dx_\mu = g_{\mu\nu} dx^\nu$ we see that

$g_{\mu\nu}$ lowers indices of 4-vectors, tensors, etc.

Example: $\underbrace{x^\mu = (ct, \vec{x})}_{\text{contravariant}}$, $\underbrace{x_\mu = g_{\mu\nu} x^\nu = (ct, -\vec{x})}_{\text{covariant}}$.

Def. g is an inverse [illegible]

$$A_\mu = g_{\mu\nu} A^\nu \Rightarrow A^\mu = g^{\mu\nu} A_\nu$$

invert $g_{\mu\nu}$ matrix

where $\delta^\mu{}_\nu = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$.

(e.g. $x_\mu = g_{\mu\nu} x^\nu, \ldots$)

$\Rightarrow g^{\mu\nu}$ raises indices of 4-vetors, tensors, etc.

$$\delta^\mu{}_\nu = g^{\mu\alpha} \cdot g_{\alpha\nu} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \Rightarrow g^{\mu\nu} = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix} = g_{\mu\nu}$$

on Minkowski space!

indeed, as $A_\mu B^\mu = \delta^\mu{}_\nu \cdot A_\mu B^\nu = g^\mu{}_\nu A_\mu B^\nu$.

Define an abbreviated notation: $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$$

$\Rightarrow \partial_\mu \varphi$ is a covariant vector

($\partial^\mu \varphi$ is a contravariant vector (check!))

$$\left(\frac{\partial \varphi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \varphi}{\partial x^\nu} \right)$$

$\partial_\mu A^\mu$ is Lorentz-invariant

4d Laplace operator $\dfrac{\partial^2}{c^2 \partial t^2} - \vec{\nabla}^2 = \partial_\mu \partial^\mu$ is $\left(= \dfrac{\partial}{\partial x^0}\dfrac{\partial}{\partial x_0} + \dfrac{\partial}{\partial x^2}\dfrac{\partial}{\partial x_2} \right)$

(d'Alembertian)

also Lorentz-invariant. (can you prove this?)

4 - velocity

Let's define a 4-vector for velocity:

$$dx^\mu = (dx^0, dx^1, dx^2, dx^3) \Rightarrow v^\mu \stackrel{?}{=} \frac{dx^\mu}{dt} \; ?$$

But: time is not a scalar!

$\frac{dx^\mu}{dt} \sim \frac{dx^\mu}{dx^0}$ ~ not a Lorentz-vector.

$\Rightarrow$ try proper time $d\tau = \frac{ds}{c} \Rightarrow u^\mu \equiv \frac{dx^\mu}{d\tau}$ 4-velocity.

as $d\tau = \frac{dt}{\gamma} \Rightarrow u^0 = \frac{c\,dt}{dt/\gamma} = c\gamma$

$\vec{u} = \frac{d\vec{x}}{dt} \cdot \gamma = \gamma \cdot \vec{v} \Rightarrow \boxed{u^\mu = \gamma(c, \vec{v})}$.

Note that $u_\mu u^\mu = c^2$.

Boost in terms of rapidity.

$$\begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$ ~ boost along the z-axis

$\Rightarrow$ Define rapidity η

by $\boxed{\beta \equiv \tanh \eta} = \frac{e^\eta - e^{-\eta}}{e^\eta + e^{-\eta}}$

Define light-cone coordinates $A^+ = \frac{A^0 + A^3}{\sqrt{2}}$

$A^- = \frac{A^0 - A^3}{\sqrt{2}}$

$\Rightarrow$ then

$A_\mu A^\mu = 2A^+A^- - (A^1)^2 - (A^2)^2$

$$A'^{+} = \frac{1}{\sqrt{2}}\left(A'^{0} + A'^{3}\right) = \frac{1}{\sqrt{2}}\gamma\left(A^{0} - \beta A^{3} - \beta A^{0} + A^{3}\right) =$$

$$= \frac{1}{\sqrt{2}}\gamma(1-\beta)\left(A^{0} + A^{3}\right) = \gamma(1-\beta)A^{+}$$

$$A'^{-} = \frac{1}{\sqrt{2}}\left(A'^{0} - A'^{3}\right) = \gamma(1+\beta)A^{-}$$

$\gamma(1-\beta)\cdot\gamma(1+\beta) = 1 \Rightarrow$ define $\gamma(1-\beta) = e^{-\eta} \Rightarrow$

$\gamma(1+\beta) = e^{+\eta} \Rightarrow \dfrac{1-\beta}{1+\beta} = e^{-2\eta} \Rightarrow \dfrac{1-e^{-2\eta}}{1+e^{-2\eta}} = \beta \Rightarrow$

$\Rightarrow \beta = \tanh\eta$. as expected!

$\Rightarrow$ Rapidity makes boosts easy!

$$\boxed{A'^{+} = e^{-\eta}A^{+}\ ; \quad A'^{-} = e^{\eta}A^{-}\ ; \quad A'^{1,2} = A^{1,2}}$$

$-\infty < \eta < +\infty, \Rightarrow -1 < \beta = \tanh\eta < +1$

Two boosts: $A''^{+} = e^{-\eta_2}A'^{+} = e^{-\eta_1-\eta_2}A^{+}$

$$A''^{-} = e^{\eta_1+\eta_2}A^{-}$$

$\sim$ a simple addition of rapidities!