Last time

4-velocity: $u^\mu \equiv \frac{dx^\mu}{d\tau}$ ← proper time

$\Rightarrow$ $\boxed{u^\mu = \gamma(c, \vec{v})}$ $\Rightarrow$ $\boxed{u_\mu u^\mu = c^2}$

3-velocity

Boosts in terms of rapidity

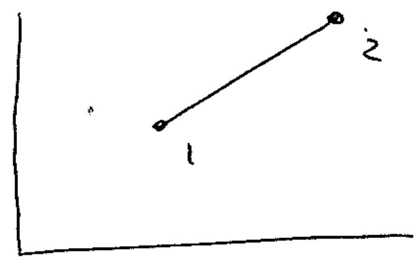
A^μ : $A^\pm = \frac{A^0 \pm A^3}{\sqrt{2}}$ $\Rightarrow$ boosts along the z-axis

are
$$\begin{cases} A^+ \to e^{-\eta} A^+ \\ A^- \to e^{\eta} A^- \\ A^{1,2} \to A^{1,2} \end{cases}$$
where $\boxed{\beta = \tanh \eta}$.

$\Rightarrow$ $-\infty < \eta < +\infty$ $\Rightarrow$ $-1 < \beta < +1$.

Relativistic Mechanics

Consider a free particle (moving along a straight line). We need to construct a Lorentz-invariant action for such a particle. It's characterized by a 4-vector x^μ $\Rightarrow$ the



only Lorentz-invariant is the interval $\Rightarrow$

$\Rightarrow$ action should be a function of m & $\vec{v}$.

$\Rightarrow \int_1^2 ds$ (can't have $\int_1^2 (ds)^2$ ~ still infinitesimal, can't have $\int ds\, f(x_\mu x^\mu)$ $\Rightarrow$ can not depend on position, $\int ds\, f(u_\mu u^\mu) = \int ds\, \underbrace{f(c^2)}_{\text{const.}}$ $\Rightarrow$ dependence on u^μ is trivial

$\Rightarrow$ write the action S as $S = -A \cdot \int_1^2 ds$

As $ds^2 = c^2 dt^2 - (d\vec{x})^2 = c^2 dt^2 \left(1 - \beta^2(t)\right)$ $\Rightarrow$

$\Rightarrow ds = c\,dt\sqrt{1-\beta^2(t)}$ $\Rightarrow$ $S = -Ac\int_{t_1}^{t_2} dt\,\sqrt{1-\beta^2(t)}$.

$\Rightarrow$ as $S = \int_{t_1}^{t_2} dt \cdot L$, where L is the Lagrangian,

$$\Rightarrow L = -AC\sqrt{1-\beta^2(t)} \, .$$

Now, in classical non-relativistic mechanics we know that $L = K - V$

kinetic energy, potential energy

$\Rightarrow$ for a free NR particle $V = 0 \Rightarrow L = K = \frac{1}{2} m v^2$

(We know that in non-relativistic (NR) limit: $K = \frac{1}{2} m v^2$) $\Rightarrow$ as $\beta \to 0 \Rightarrow$

$$\Rightarrow L = \underbrace{-AC}_{\text{constant} \sim \text{drop, not important for dynamics}} + Ac \frac{1}{2} \beta^2 + \ldots$$

$$\Rightarrow A \cancel{c} \frac{\cancel{1}}{\cancel{2}} \frac{\cancel{v^2}}{c^{\cancel{2}}} = \frac{\cancel{1}}{\cancel{2}} m \cancel{v^2} \Rightarrow A = mc$$

$$\Rightarrow \boxed{S = -mc \int_1^2 ds = -mc^2 \int_{t_1}^{t_2} dt \sqrt{1-\beta^2(t)}}$$

$$\boxed{L = -mc^2\sqrt{1-\beta^2(t)}}$$

action and Lagrangian for a free point particle

The particle's <u>Energy & Momentum.</u>

The free particle's degrees of freedom are coordinates $\vec{x}$ & t. <u>Momentum</u> is defined by: $p^i = \dfrac{\partial L}{\partial \dot{x}^i}$, where $i = 1, 2, 3$ and $\dot{x}^i = \dfrac{dx^i}{dt}$.
(sign = convention)
(know from classical mechanics).

$$\Rightarrow \quad p^i = \frac{\partial L}{\partial v^i} = -mc^2 \frac{-\cancel{2} v^i / c^2}{\cancel{2}\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m \vec{v}$$

(cf. Hamiltonian)
<u>Energy</u> is defined by $E = \vec{p} \cdot \dot{\vec{x}} - L =$

$$= \vec{p} \cdot \vec{v} - L = \gamma m v^2 + mc^2 \sqrt{1 - \frac{v^2}{c^2}} =$$

$$= \gamma \left[\cancel{mv^2} + mc^2 \left(1 - \cancel{\frac{v^2}{c^2}}\right) \right] = mc^2 \gamma$$

$$\Rightarrow \quad E = mc^2 \gamma = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$ (world's most famous formula)

Remember that $u^\mu = \gamma(c, \vec{v})$ a 4-velocity.

We now see that $\left(\frac{E}{c}, \vec{p}\right) = m\gamma(c, \vec{v}) \Rightarrow$

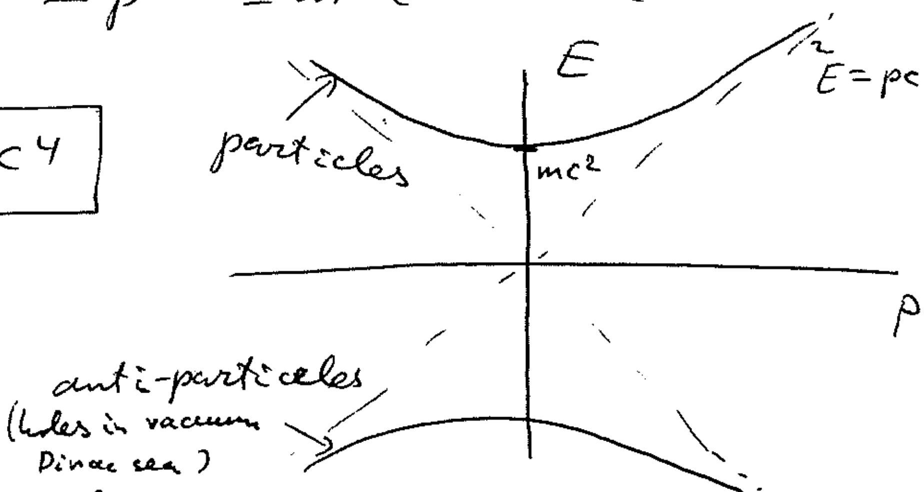
$\Rightarrow$ we have a new <u>4-momentum</u> four-vector:

$$\boxed{p^\mu = m u^\mu}, \quad \text{where } p^0 = \frac{E}{c}, \quad p^i = (\vec{p})^i$$

Note that $p_\mu p^\mu = m^2 u_\mu u^\mu = m^2 \gamma^2 (c^2 - v^2) =$

$= m^2 c^2 \Rightarrow \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2 c^2$ or

$$\boxed{E^2 = \vec{p}^{\,2} c^2 + m^2 c^4}$$



4-vector p^μ transforms in the usual way:

$$\boxed{\begin{aligned} p^0 &= \gamma(p^{0\prime} + \beta p^{x\prime}) \\ p^x &= \gamma(p^{x\prime} + \beta p^{0\prime}) \\ p^y &= p^{y\prime}, \quad p^z = p^{z\prime} \end{aligned}}$$

(boost in x-direction)

Def.

<u>kinetic energy</u>

$$T = E(v) - E(0) = mc^2[\gamma_u - 1]$$

Energy Conservation

$L = L(q, \dot{q}) \Rightarrow$ under $t \to t + \Delta t$ the Lagrangian changes as

$$\frac{dL}{dt}\Delta t = \Delta t \left[\frac{\delta L}{\delta q}\dot{q} + \frac{\delta L}{\delta \dot{q}}\ddot{q} \right]$$

Use equations of motion (EOM)

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\delta \dot{q}}\right) = 0}$$

to write

$$\frac{dL}{dt}\Delta t = \Delta t \left[\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\right)\dot{q} + \frac{\delta L}{\delta \dot{q}}\ddot{q} \right] = \Delta t \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\dot{q}\right)$$

$$\Rightarrow \frac{d}{dt}\left[\frac{\delta L}{\delta \dot{q}}\dot{q} - L \right] = 0$$

$\Rightarrow$ Def. Hamiltonian as

$$\boxed{H(p, q) \equiv p\dot{q} - L}$$ where $p = \frac{\delta L}{\delta \dot{q}}$ ~ canonical momentum

$\Rightarrow$ identify it with the energy $\Rightarrow$

$$E = \frac{\delta L}{\delta \dot{q}}\dot{q} - L = p\dot{q} - L \Rightarrow \frac{dE}{dt} = 0$$

$\Rightarrow$ energy is conserved!

Newtonian mechanics: $\vec{F} = \frac{d\vec{p}}{dt}$ (force)

$\Rightarrow$ define force as $f^\mu = \frac{dp^\mu}{d\tau}$

$\Rightarrow \vec{f} = \frac{d\vec{p}}{dt}\gamma \Rightarrow$ in NR limit gives Newtonian result.

"$\vec{F}\cdot\gamma$

$$\frac{dp^0}{d\tau} = \gamma\frac{dp^0}{dt};$$ Note that $f^\mu u_\mu = 0$

$$\left(u_\mu f^\mu = u_\mu \frac{dp^\mu}{d\tau} = u_\mu m \frac{du^\mu}{d\tau} = \frac{1}{2} m \frac{d(u_\mu u^\mu)}{d\tau} = \right.$$

$$\left. = \frac{1}{2} m \frac{dc^2}{d\tau} = 0 \right) \Rightarrow f^0 \cdot u^0 = \vec{f}\cdot \gamma\vec{V} \Rightarrow$$

$$\Rightarrow f^0 c = \vec{f}\cdot\vec{V} \Rightarrow f^0 = \frac{\vec{f}\cdot\vec{V}}{c} \Rightarrow \gamma\frac{dp^0}{dt} = f^0 = \frac{\vec{f}\cdot\vec{V}}{c}$$

$$\Rightarrow \gamma\frac{dE}{dt} = \vec{f}\cdot\vec{V} = \gamma\vec{F}\cdot\vec{V} \Rightarrow \frac{dE}{dt} = \vec{F}\cdot\vec{V}$$

($\vec{F}$ is Newtonian NR force).

$\Rightarrow$ 4-momentum is conserved in particle interactions.

$$\boxed{\sum p^\mu_{\text{initial}} = \sum p^\mu_{\text{final}}}$$

Particle Decay

- Imagine a particle with mass M at rest which decays into 2 particles with masses m_1 & m_2:

$$\underset{M}{\cdot} \Longrightarrow \overset{\longleftarrow}{m_1} \quad \overset{\longrightarrow}{m_2}$$

$$p^\mu = p_1^\mu + p_2^\mu \text{, where } p^\mu = (Mc, \vec{0})$$

$$p_1^\mu = \left(\frac{E_1}{c}, \vec{p}_1\right), \quad p_2^\mu = \left(\frac{E_2}{c}, \vec{p}_2\right)$$

$\Rightarrow$ $\mu = 0$ $\Rightarrow$ energy conservation $\Rightarrow$

$$Mc = \frac{E_1}{c} + \frac{E_2}{c}$$

$\mu = i$ $\Rightarrow$ momentum conservation: $\vec{p}_1 + \vec{p}_2 = 0$

Rewrite $p^\mu - p_1^\mu = p_2^\mu$ $\Rightarrow$ square $\Rightarrow$

$$(p - p_1)^2 = p_2^2 = m_2^2 c^2 \Rightarrow p^2 + p_1^2 - 2 p \cdot p_1 = m_2^2 c^2$$

$$\Rightarrow M^2 c^2 + m_1^2 c^2 - 2Mc \cdot \frac{E_1}{c} = m_2^2 c^2$$

$$\Rightarrow E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2 \Rightarrow$$

as $E_1 > m_1 c^2$, $E_2 > m_2^2 c^2$

$\Rightarrow$ $M > m_1 + m_2$

otherwise decay can't happen.

similarly

$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} c^2 .$$