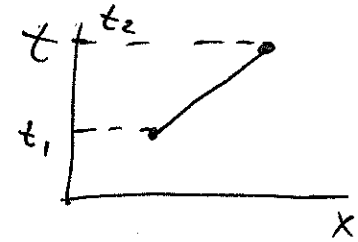
Last time | Relativistic Mechanics

for a free point particle we wrote the action

$$\boxed{S = -mc \int_1^2 ds = -mc^2 \int_{t_1}^{t_2} dt \sqrt{1-\beta^2(t)}}$$

and the Lagrangian



$$\boxed{L = -mc^2 \sqrt{1-\beta^2(t)}}$$

Used canonical momentum $p^i = \frac{\partial L}{\partial \dot{x}^i}$ to get

$$\boxed{\vec{p} = m\gamma\vec{v}}$$

Energy was derived from the Hamiltonian:

$$\boxed{E = m\gamma c^2} \quad \Leftarrow \quad E = \vec{p}\cdot\dot{\vec{x}} - L$$

Note that E and $\vec{p}$ form a 4-vector of momentum

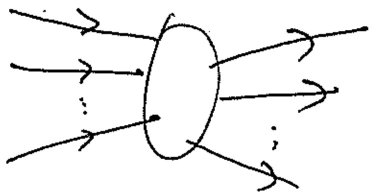
$$\boxed{p^\mu = m u^\mu = \left(\frac{E}{c}, \vec{p}\right)}.$$

$$E^2 = \vec{p}^{\,2} c^2 + m^2 c^4$$ ~ another useful result

Four-vector of force: $\boxed{f^\mu = \frac{dp^\mu}{d\tau} = \gamma\left(\frac{\vec{F}\cdot\vec{v}}{c}, \vec{F}\right)}$

where $\boxed{\vec{F} = \frac{d\vec{p}}{dt}}$ ~ usual Newtonian force.

$\Rightarrow$ we got $\boxed{\frac{dE}{dt} = \vec{F}\cdot\vec{v}}$, also like in Newtonian mechanics



$$\boxed{\sum_{\text{particles}} p^\mu_{\text{init}} = \sum_{\text{particles}} p^\mu_{\text{final}}}$$

energy + momentum conservation

Newtonian mechanics: $\vec{F} = \frac{d\vec{p}}{dt}$ (force)

$\Rightarrow$ define force as $f^\mu = \frac{dp^\mu}{d\tau}$

$\Rightarrow \vec{f} = \frac{d\vec{p}}{dt}\gamma \Rightarrow$ in NR limit gives
"$\vec{F}\cdot\gamma$ Newtonian result.

$\frac{dp^0}{d\tau} = \gamma \frac{dp^0}{dt}$; Note that $f^\mu u_\mu = 0$

$\left(u_\mu f^\mu = u_\mu \frac{dp^\mu}{d\tau} = u_\mu m \frac{du^\mu}{d\tau} = \frac{1}{2} m \frac{d(u_\mu u^\mu)}{d\tau} = \right.$

$\left. = \frac{1}{2} m \frac{dc^2}{d\tau} = 0 \right) \Rightarrow f^0 \cdot u^0 = \vec{f}\cdot \gamma\vec{v} \Rightarrow$

$\Rightarrow f^0 c = \vec{f}\cdot\vec{v} \Rightarrow f^0 = \frac{\vec{f}\cdot\vec{v}}{c} \Rightarrow \gamma \frac{dp^0}{dt} = f^0 = \frac{\vec{f}\cdot\vec{v}}{c}$

$\Rightarrow \gamma \frac{dE}{dt} = \vec{f}\cdot\vec{v} = \gamma \vec{F}\cdot\vec{v} \Rightarrow \frac{dE}{dt} = \vec{F}\cdot\vec{v}$

($\vec{F}$ is Newtonian NR force).

$\Rightarrow$ 4-momentum is conserved in particle interactions.

$$\boxed{\sum p^\mu_{initial} = \sum p^\mu_{final}}$$

Particle Decay

Imagine a particle with mass M at rest which decays into 2 particles with masses m_1 & m_2:

$$\underset{M}{\cdot} \Rightarrow \overset{\longleftarrow}{m_1} \quad \overset{\longrightarrow}{m_2}$$

$$p^\mu = p_1^\mu + p_2^\mu \text{, where } p^\mu = (Mc, \vec{0})$$

$$p_1^\mu = \left(\frac{E_1}{c}, \vec{p}_1\right), \quad p_2^\mu = \left(\frac{E_2}{c}, \vec{p}_2\right)$$

$\Rightarrow \mu = 0 \Rightarrow$ energy conservation $\Rightarrow$

$$Mc = \frac{E_1}{c} + \frac{E_2}{c}$$

$\mu = i \Rightarrow$ momentum conservation: $\vec{p}_1 + \vec{p}_2 = 0$

Rewrite $p^\mu - p_1^\mu = p_2^\mu \Rightarrow$ square $\Rightarrow$

$$(p - p_1)^2 = p_2^2 = m_2^2 c^2 \Rightarrow p^2 + p_1^2 - 2p \cdot p_1 = m_2^2 c^2$$

$$\Rightarrow M^2c^2 + m_1^2c^2 - 2Mc \cdot \frac{E_1}{c} = m_2^2 c^2$$

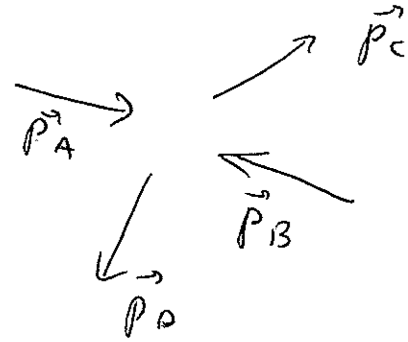
$$\Rightarrow E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2 \Rightarrow \text{as } E_1 > m_1 c^2,\ E_2 > m_2^2 c^2$$

$\Rightarrow M > m_1 + m_2$ otherwise decay can't happen.

similarly
$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} c^2 .$$

Particle Scattering

Imagine particles A & B colliding to become particles B & C:



$$p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu .$$

$$\Rightarrow \quad \vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D$$

$$E_A + E_B = E_C + E_D$$

<u>Invariant mass</u> of the particles: $M^2 = \left(\sum_i p_i^\mu\right)^2 / c^2$

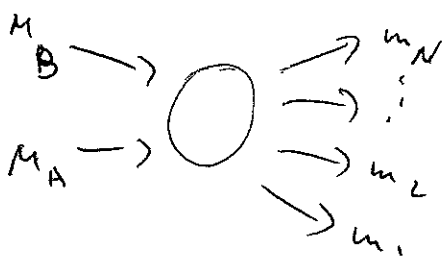
e.g. $M^2 = (p_A + p_B)^2 / c^2$.

also known as ceter-of-mass energy : $s = (p_A + p_B)^2$.

(Go to the center-of-mass frame where

$\vec{p}_A{}' + \vec{p}_B{}' = 0 \Rightarrow (p_A + p_B)^2 = \left(\frac{E_1'}{c} + \frac{E_2'}{c}\right)^2$ – only energy in CMS frame.)

<u>Threshold energy</u>:



$$\Rightarrow (p_A + p_B)^2 = s = (p_1 + \ldots + p_N)^2 =$$

$$= (E_1' + \ldots + E_N')^2 \frac{1}{c^2} \geq \frac{(m_1 + \ldots + m_N)^2 c^4}{c^2}$$

CMS frame

$\Rightarrow$ minimum energy squared for a process to take place is $$S_{min} = (m_1 + m_2 + \dots + m_N)^2 c^2 .$$

This is known as threshold energy.