Relativistic Particles in Electromagnetic Fields

The Lagrangian and the action

Let us construct the Lagrangian for the interaction of electromagnetic field with a point particle of charge $e$ and mass $m$.

The Lagrangian for the particle is

$$L_{\text{free}} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}.$$

Given the action for a system

$$S = \int_{t_1}^{t_2} dt \ L\left(\xi^i(t), \dot{\xi}^i(t), t\right)$$

the least action principle gives us equations of motion (Euler-Lagrange equations)

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\xi}^i} \right) - \frac{\partial L}{\partial \xi^i} = 0$$

$q^i$ are generalized coordinates, $\dot{q}^i = \frac{dq^i}{dt}$ velocities.

Apply this to $L_{\text{free}}$: $q^i(t) = x^i(t)$,

$$\dot{q}^i(t) = \frac{d}{dt} x^i(t) = \dot{x}^i(t) = \frac{v^i}{c} \Rightarrow L_{\text{free}} = L\left(\dot{x}^i\right)$$

$$\Rightarrow \frac{\partial L_{\text{free}}}{\partial x^i} = 0, \quad \frac{\partial L}{\partial \dot{x}^i} = -mc^2 \frac{-2v^i/c^2}{\sqrt{1 - v^2/c^2}} = m \delta v^i$$

momentum

$$\Rightarrow \text{EOM are } \frac{d}{dt} (m \delta \dot{v}) = 0 \Rightarrow \frac{d\tilde{p}}{dt} = 0 \Rightarrow \tilde{p} = \text{const}$$
Electromagnetic field: imagine a 4-vector field \( A^\mu(x) \), where \( x \) denotes a point in 4-dim space-time \( x^\mu \).

\[
A^\mu(x) = (A^0(x), \vec{A}(x))
\]

**Def.** \( A^0(x) = \Phi(x) \) is the scalar potential

**Def.** \( \vec{A}(x) = (A'^1(x), A'^2(x), A'^3(x)) \) is the vector potential

We need to construct a term in the action (Lagrangian) which combines both the field \( A^\mu(x) \) and the motion of the particle, described by \( x^\mu(t) \) in a trajectory.

Physics is Lorentz-invariant \( \Rightarrow \) so is the action. To get a \( L \)-inv. object multiply the field \( A^\mu \) by the infinitesimal element of the particle's world line \( dx_\mu \):

\[
S \sim \int_{x^\mu(t)}^{x^\mu(\tau)} A^\mu \, dx_\mu.
\]

The integral goes along the particle's trajectory. (Took \( S \) to be linear in \( A^\mu \) = experimental result. Std \( A^\mu \) allowed...
It turns out that the proportionality coefficient is \(-\frac{e}{c}\), such that

\[
S_{\text{int}} = -\frac{e}{c} \int \frac{1}{c} \int dx^\mu A^\mu.
\]

\(c = \) speed of light, \(e = \) particle's charge
(naturally neutral particles do not feel E\&M field \(\Rightarrow\) if \(e = 0\) need \(S_{\text{int}} = 0\))

\[
S_{\text{int}} = -\frac{e}{c} \int \frac{1}{c} \int dx^\mu A^\mu = -\frac{e}{c} \int \frac{1}{c} \int dx^\mu \frac{d}{dt} \frac{1}{8} U_\mu A^\mu
\]

where \(U^\mu = \frac{dx^\mu}{dt}\) is the 4-velocity.

Since \(S = \int dt \, L \Rightarrow\)

\[
L_{\text{int}} = -\frac{e}{c} \frac{1}{8} U_\mu A^\mu.
\]

The total Lagrangian for particle in the E\&M field is

\[
L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{e}{c} q \ U_\mu A^\mu \Rightarrow \quad \text{as} \quad U^\mu = \delta(c, \vec{v})
\]

\[
L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - e \Phi + \frac{e}{c} \vec{v} \cdot \vec{A}
\]

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familiar from electrostatics term

\(\Rightarrow\) could be used to fix the prefactor in \(S_{\text{int}}\).
Equations of motion: \[ \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial x^i} = 0 \]

3) \[ \frac{\partial L}{\partial v^i} = m \frac{\partial v^i}{\partial v^i} + \frac{e}{c} A^i = \dot{p}^i \text{ canonical momentum} \]

\[ \hat{\dot{p}} = \dot{p} + \frac{e}{c} \dot{A} \]

\[ \frac{\partial L}{\partial x^i} = -e \frac{\partial \Phi}{\partial x^i} + \frac{e}{c} \dot{v} \frac{\partial A^i}{\partial x^i} \]

\[ \Rightarrow \frac{d}{dt} \left( m \frac{\partial v^i}{\partial x^i} + \frac{e}{c} A^i \right) + e \frac{\partial \Phi}{\partial x^i} - \frac{e}{c} \dot{v} \frac{\partial A^i}{\partial x^i} = 0 \]

\[ \frac{dp^i}{dt} = -e \frac{\partial \Phi}{\partial x^i} - \frac{e}{c} \left[ \frac{dA^i}{dt} - \dot{v} \frac{\partial A^i}{\partial x^i} \right] \]

As \[ \frac{dA^i}{dt} = \frac{\partial A^i}{\partial t} + \frac{\partial x^i}{\partial t} \frac{\partial A^i}{\partial x^i} = \frac{\partial A^i}{\partial t} + \dot{v} \frac{\partial A^i}{\partial x^i} \]

we get

\[ \frac{dp^i}{dt} = -e \frac{e}{c} \left[ c \left( \frac{\partial A^i}{\partial x^i} + \frac{\partial A^i}{\partial x^i} \right) \right] \]

Note that \[ \frac{\partial \Phi}{\partial x^i} = m \frac{\partial \Phi}{\partial x^i} \text{ and } x^0 = ct \Rightarrow \]

\[ \frac{dp^i}{dt} = -e \frac{e}{c} \left[ c \left( -\frac{\partial A^i}{\partial x_i} + \frac{\partial A^i}{\partial x^o} \right) - \dot{v} \left( \frac{\partial A^i}{\partial x_i} + \frac{\partial A^i}{\partial x_j} \right) \right] \]
Euler–Lagrange Equations

\[ S = \int_{t_1}^{t_2} dt \ L (q, \dot{q}) \]

\[ \Rightarrow \text{least action principle requires that } S S = 0 \]

under \[ q \to q + \delta q \]. We get

\[ 0 = SS' = \left[ \int_{t_1}^{t_2} dt \left[ \frac{\delta L}{\delta q} \delta q + \frac{\delta L}{\delta \dot{q}} \delta \dot{q} \right] \right] = \left. \frac{\delta L}{\delta q} \delta q \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left[ \frac{\delta L}{\delta \dot{q}} - \frac{d}{dt} \left( \frac{\delta L}{\delta \ddot{q}} \right) \right] \delta q \]

\[ \Rightarrow \text{require that } \delta q (t_1) = \delta q (t_2) = 0 \Rightarrow \text{drop the first term} \]

\[ \Rightarrow \text{since the above result is true for any } \delta q (t) \]

\[ \Rightarrow \frac{\delta L}{\delta \dot{q}} - \frac{d}{dt} \left( \frac{\delta L}{\delta \ddot{q}} \right) = 0 \]

Euler–Lagrange equation

aka

equation of motion (EOM)