Relativistic Particles in Electromagnetic Fields

The Lagrangian and the action

Let us construct the Lagrangian for the interaction of electromagnetic field with a point particle of charge e and mass m.

The Lagrangian for the particle is

$$L_{free} = -mc^2 \sqrt{1-\frac{V^2}{c^2}}.$$

Given the action for a system

$$S = \int_{t_1}^{t_2} dt \, L(q_i(t), \dot{q}_i(t), t)$$

the least action principle gives us equations of motion (EOM) (Euler-Lagrange equations)

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$

q_i ~ generalized coordinates, $\dot{q}_i = \frac{dq_i}{dt}$ ~ velocities

apply this to L_{free}: $q_i(t) = x^i(t)$,

$$\dot{q}_i(t) = \frac{d}{dt} x^i(t) = \dot{x}^i(t) = V^i(t) \Rightarrow L_{free} = L_{free}(\dot{x}^i)$$

$$\Rightarrow \frac{\partial L_{free}}{\partial x^i} = 0, \quad \frac{\partial L}{\partial \dot{x}^i} = -mc^2 \frac{-2V^i/c^2}{2\sqrt{1-\frac{V^2}{c^2}}} = m\gamma V^i$$

momentum conservation

$$\Rightarrow \text{EOM are} \quad \frac{d}{dt}(m\gamma \vec{V}) = 0 \Rightarrow \frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p} = \text{const}$$

Electromagnetic field: imagine a 4-vector field $A_\mu(x)$, where x denotes a point in 4-dim space-time x^μ.

$$A^\mu(x) = \left(A^0(x), \vec{A}(x) \right)$$

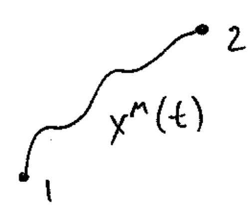
Def. $A^0(x) \equiv \Phi(x)$ is the <u>scalar potential</u>

Def. $\vec{A}(x) = \left(A^1(x), A^2(x), A^3(x) \right)$ is the <u>vector potential</u>.

We need to construct a term in the action (Lagrangian) which combines both the field $A^\mu(x)$ and the motion of the particle, described by $x^\mu(t)$ ~ a trajectory.

Physics is Lorentz-invariant $\Rightarrow$ so is the action. To get a L-inv. object multiply the field A^μ by the infinitesimal element of the particle's world line (dx_μ):

$$S_{int} \sim \int_1^2 A^\mu \, dx_\mu \, .$$



The integral goes along the particle's trajectory.
(Took S to be linear in A^μ ~ experimental result. $\int ds \, A_\mu A^\mu$ ~ allowe.

It turns out that the proportionality coefficient is $-\frac{e}{c}$, such that

$$\boxed{S_{int} = -\frac{e}{c}\int_1^2 dx_\mu A^\mu}$$

c = speed of light, e = particle's charge
(naturally neutral particles do not feel E&M field $\Rightarrow$ if $e=0$ need $S_{int}=0$)

$$S_{int} = -\frac{e}{c}\int_{t_1}^{t_2} dt\,\frac{dx_\mu}{dt}A^\mu \underset{\substack{\uparrow \\ \text{as } dt = \gamma d\tau}}{=} -\frac{e}{c}\int_{t_1}^{t_2} dt\,\frac{1}{\gamma}u_\mu A^\mu$$

where $u^\mu = \frac{dx^\mu}{d\tau}$ is the 4-velocity.

Since $S = \int dt\, L \Rightarrow \boxed{L_{int} = -\frac{e}{c}\frac{1}{\gamma}u_\mu A^\mu}$.

The total Lagrangian for particle in the E&M field is

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - \frac{e}{c\gamma}u_\mu A^\mu \Rightarrow \Big| \text{ as } u^\mu = \gamma(c, \vec{v})$$

$$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - \underbrace{e\Phi}_{\text{familiar from electrostatics term}} + \frac{e}{c}\vec{v}\cdot\vec{A}}$$

$\Rightarrow$ could be used to fix the prefactor in S_{int}

Equations of motion: $\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} = 0$

$$\Rightarrow \quad \frac{\partial L}{\partial v^i} = m\gamma v^i + \frac{e}{c}A^i = P^i \sim \underline{\text{canonical momentum}}$$

$$\boxed{\vec{P} = \vec{p} + \frac{e}{c}\vec{A}}$$

$$\frac{\partial L}{\partial x^i} = -e\frac{\partial \Phi}{\partial x^i} + \frac{e}{c}v^j\frac{\partial A^j}{\partial x^i}$$

$$\Rightarrow \quad \frac{d}{dt}\Big(\underbrace{m\gamma v^i}_{p^i} + \frac{e}{c}A^i\Big) + e\frac{\partial \Phi}{\partial x^i} - \frac{e}{c}v^j\frac{\partial A^j}{\partial x^i} = 0$$

$$\frac{dp^i}{dt} = -e\frac{\partial \Phi}{\partial x^i} - \frac{e}{c}\left[\frac{dA^i}{dt} - v^j\frac{\partial A^j}{\partial x^i}\right]$$

As $\frac{dA^i(x)}{dt} = \frac{\partial A^i(x)}{\partial t} + \frac{\partial x^j}{\partial t}\frac{\partial A^i(x)}{\partial x^j} = \frac{\partial A^i}{\partial t} + v^j\frac{\partial A^i}{\partial x^j}$

we get

$$\frac{dp^i}{dt} = -\frac{e}{c}\left[c\left(\frac{\partial \overset{\Phi}{\overset{\shortparallel}{A^0}}}{\partial x^i} + \frac{1}{c}\frac{\partial A^i}{\partial t}\right) + v^j\frac{\partial A^i}{\partial x^j} - v^j\frac{\partial A^j}{\partial x^i}\right]$$

Note that $\frac{\partial}{\partial x^i} = -\frac{\partial}{\partial x_i}$ and $x^0 = ct \Rightarrow$

$$\frac{dp^i}{dt} = -\frac{e}{c}\left[c\left(-\frac{\partial A^0}{\partial x_i} + \frac{\partial A^i}{\partial x^0}\right) - v^j\left(-\frac{\partial A^j}{\partial x_i} + \frac{\partial A^i}{\partial x_j}\right)\right]$$

Euler-Lagrange Equations

$$S = \int_{t_1}^{t_2} dt\, L(q, \dot{q})$$

$\Rightarrow$ least action principle requires that $\delta S = 0$

under $q \to q + \delta q$. We get

$$0 = \delta S = \int_{t_1}^{t_2} dt \left[\frac{\delta L}{\delta q} \delta q + \frac{\delta L}{\delta \dot{q}} \underbrace{\delta \dot{q}}_{\frac{d}{dt}\delta q \Rightarrow \text{parts}} \right] = \frac{\delta L}{\delta \dot{q}} \delta q \Big|_{t_1}^{t_2} +$$

$$+ \int_{t_1}^{t_2} dt \left[\frac{\delta L}{\delta q} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}} \right) \right] \delta q$$

$\Rightarrow$ require that $\delta q(t_1) = \delta q(t_2) = 0$ $\Rightarrow$ drop the first term

$\Rightarrow$ since the above result is true for any $\delta q(t)$

$$\Rightarrow \quad \boxed{\frac{\delta L}{\delta q} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}} \right) = 0}$$

Euler-Lagrange equation
aka
equation of motion (EOM)