Last time

Relativistic Particles in Electromagnetic Fields

The Lagrangian and the action

$$L_{free} = -mc^2\sqrt{1-\frac{v^2}{c^2}} \quad \sim \text{free particle}$$

EOM: $$\boxed{\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\right) - \frac{\delta L}{\delta q} = 0}.$$

$\Rightarrow$ for L_{free} this gives $\boxed{\frac{d\vec{p}}{dt} = 0}$ with $\vec{p} = m\gamma\vec{v}$.

Def. Electromagnetic field: $A^\mu(x) = \left(A^0(x), \vec{A}(x)\right)$

$A^0(x) \equiv \Phi(x)$ ~ "scalar potential"

$\vec{A}(x) = \left(A^1(x), A^2(x), A^3(x)\right)$ ~ vector potential

$$\boxed{S_{int} = -\frac{e}{c}\int_1^2 dx_\mu A^\mu.}$$

$A^\mu(x)$ couples to the world-line of the particle

e = electric charge

It turns out that the proportionality coefficient is $-\frac{e}{c}$, such that

$$\boxed{S_{int} = -\frac{e}{c}\int_1^2 dx_\mu A^\mu}$$

c = speed of light, e = particle's charge (naturally neutral particles do not feel E&M field $\Rightarrow$ if $e=0$ need $S_{int}=0$)

$$S_{int} = -\frac{e}{c}\int_{t_1}^{t_2} dt\,\frac{dx_\mu}{dt}A^\mu \underset{\substack{\uparrow \\ \text{as } dt=\gamma d\tau}}{=} -\frac{e}{c}\int_{t_1}^{t_2} dt\,\frac{1}{\gamma}u_\mu A^\mu$$

where $u^\mu = \frac{dx^\mu}{d\tau}$ is the 4-velocity.

Since $S = \int dt\, L \Rightarrow \boxed{L_{int} = -\frac{e}{c}\frac{1}{\gamma}u_\mu A^\mu}$.

The total Lagrangian for particle in the E&M field is

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - \frac{e}{c\gamma}u_\mu A^\mu \Rightarrow \Big| \text{ as } u^\mu = \gamma(c,\vec{v})$$

$$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}} \underbrace{- e\Phi}_{} + \frac{e}{c}\vec{v}\cdot\vec{A}}$$

familiar from electrostatics term

$\Rightarrow$ could be used to fix the prefactor in S_{int}

Equations of motion: $\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} = 0$

$$\Rightarrow \frac{\partial L}{\partial v^i} = m\gamma v^i + \frac{e}{c} A^i = P^i \sim \underline{\text{canonical momentum}}$$

$$\boxed{\vec{P} = \vec{p} + \frac{e}{c}\vec{A}}$$

$$\frac{\partial L}{\partial x^i} = -e\frac{\partial \Phi}{\partial x^i} + \frac{e}{c} V^j \frac{\partial A^j}{\partial x^i}$$

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{m\gamma v^i}_{p^i} + \frac{e}{c}A^i\Big) + e\frac{\partial \Phi}{\partial x^i} - \frac{e}{c} v^j \frac{\partial A^j}{\partial x^i} = 0$$

$$\frac{dp^i}{dt} = -e\frac{\partial \Phi}{\partial x^i} - \frac{e}{c}\left[\frac{dA^i}{dt} - v^j\frac{\partial A^j}{\partial x^i}\right]$$

As $\frac{dA^i(x)}{dt} = \frac{\partial A^i(x)}{\partial t} + \frac{dx^j}{dt}\frac{\partial A^i(x)}{\partial x^j} = \frac{\partial A^i}{\partial t} + v^j\frac{\partial A^i}{\partial x^j}$

we get

$$\frac{dp^i}{dt} = -\frac{e}{c}\left[c\left(\frac{\partial \overset{\Phi}{\overset{\parallel}{A^0}}}{\partial x^i} + \frac{1}{c}\frac{\partial A^i}{\partial t}\right) + v^j\frac{\partial A^i}{\partial x^j} - v^j\frac{\partial A^j}{\partial x^i}\right]$$

Note that $\frac{\partial}{\partial x^i} = -\frac{\partial}{\partial x_i}$ and $x^0 = ct$ $\Rightarrow$

$$\frac{dp^i}{dt} = -\frac{e}{c}\left[c\left(-\frac{\partial A^0}{\partial x_i} + \frac{\partial A^i}{\partial x^0}\right) - v^j\left(-\frac{\partial A^j}{\partial x_i} + \frac{\partial A^i}{\partial x_j}\right)\right]$$

Def. Electric field ($\vec{E}$) is defined as

$$\boxed{\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}} \quad , \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \nabla^i \equiv \partial_i$$

$$\vec{\nabla}\Phi = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right).$$

Such that $E^i = -\nabla^i \Phi - \frac{1}{c}\frac{\partial A^i}{\partial t} = -\frac{\partial \Phi}{\partial x^i} - \frac{1}{c}\frac{\partial A^i}{\partial t}$

$$\Rightarrow E^i = \frac{\partial A^0}{\partial x_i} - \frac{1}{c}\frac{\partial A^i}{\partial t} = \frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial x^0} \doteq E^i$$

Def. Magnetic field ($\vec{B}$) is defined as

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \Rightarrow B^i = \varepsilon^{ijk}\nabla^j A^k = \varepsilon^{ijk}\partial_j A^k$$

where $\varepsilon^{123} = +1$, $\varepsilon^{iij} = 0$, $\varepsilon^{ijk} = -\varepsilon^{jik} = \varepsilon^{jki} = \ldots$

Levi - Civita symbol

$$\boxed{\partial_\mu \equiv \frac{\partial}{\partial x^\mu}}$$

$$\Rightarrow B^i = \varepsilon^{ijk}\left[\underbrace{\frac{1}{2}\left(\partial_j A^k - \partial_k A^j\right)}_{\text{anti-symmetric}} + \underbrace{\frac{1}{2}\left(\partial_j A^k + \partial_k A^j\right)}_{\text{symmetric}}\right]$$

$A^{ij} = -A^{ji}$ ~ anti-symmetric tensor

$S^{ij} = S^{ji}$ ~ symmetric tensor

$$\Rightarrow A_{ij}S^{ij} = -A_{ji}S^{ji} = \Big/ \begin{matrix} j \to i \\ i \to j \end{matrix} = -A_{ij}S^{ij} = 0$$

$\Rightarrow$ convolution of symm. & anti-symm. tensors is zero.

$$B^i = \frac{1}{2} \varepsilon^{ijk} \left(\partial_j A^k - \partial_k A^j \right)$$

(sum over i implied)

Use $\varepsilon^{ijk} \varepsilon^{ij'k'} = \delta^{jj'} \delta^{kk'} - \delta^{jk'} \delta^{j'k}$ $\Rightarrow$

$$\Rightarrow B^i \varepsilon^{ij'k'} = \frac{1}{2} \left(\delta^{jj'} \delta^{kk'} - \delta^{jk'} \delta^{j'k} \right) \left(\partial_j A^k - \partial_k A^j \right)$$

$$= \frac{1}{2} \left(\partial_{j'} A^{k'} - \partial_{k'} A^{j'} \right) - \frac{1}{2} \left(\partial_{k'} A^{j'} - \partial_{j'} A^{k'} \right)$$

$$= \partial_{j'} A^{k'} - \partial_{k'} A^{j'}$$

$$\Rightarrow \frac{\partial A^k}{\partial x^j} - \frac{\partial A^j}{\partial x^k} = \varepsilon^{jki} B^i$$

$$\text{or} \quad \frac{\partial A^i}{\partial x_j} - \frac{\partial A^j}{\partial x_i} = \varepsilon^{ijk} B^k$$

Returning to the EOM we write:

$$\frac{dp^i}{\partial t} = -\frac{e}{c} \left[-c E^i - \underbrace{v^j \varepsilon^{ijk} B^k} \right]$$

$$\varepsilon^{ijk} v^j B^k = \left(\vec{v} \times \vec{B} \right)^i$$

$$\Rightarrow \boxed{\frac{d\vec{p}}{dt} = e \vec{E} + \frac{e}{c} \vec{v} \times \vec{B}}$$

Lorentz force!