Last time (Defined) Field-strength tensor

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

~ components are $\vec{E}$ & $\vec{B}$ fields

We combined Lorentz-force equation

$$\boxed{\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}}$$

with the energy evolution equation

$$\boxed{\frac{d\mathcal{E}}{dt} = e\,\vec{v}\cdot\vec{E}}$$

into one Lorentz-covariant expression:

$$\boxed{\frac{dp^\mu}{d\tau} = \frac{e}{c}u_\nu F^{\mu\nu}}$$

We have also constructed the Hamiltonian for a relativistic particle in external E&M field:

$$H = \underbrace{\sqrt{m^2c^4 + c^2\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}}_{\sqrt{m^2c^4 + c^2p^2}} + e\,\Phi$$

~ usual energy of a point particle

Non-relativistic limit:

$$\boxed{H \simeq mc^2 + \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m} + e\Phi}$$

~ perhaps familiar from non-relativistic QM class

$$\Rightarrow \boxed{H = \sqrt{m^2c^4 + (c\vec{P} - e\vec{A})^2} + e\Phi}$$ total energy of the particle

$\sqrt{m^2c^4 + p^2c^2}$

Motion of a point charge in external $\vec{E}, \vec{B}$ fields:

We have Lorentz force $\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)$

and energy change $\frac{dE}{dt} = q\,\vec{v}\cdot\vec{E}$.

A. Constant Uniform Electric Field.

$\frac{d\vec{p}}{dt} = q\vec{E} \Rightarrow \vec{p} = q\vec{E}t + \text{const} \Rightarrow$ if

the particle starts from rest $\Rightarrow \vec{p}|_{t=0} = 0$

$\Rightarrow \vec{p} = q\vec{E}t \Rightarrow$ is $\vec{E} = E\hat{x} \Rightarrow$

$\Rightarrow p_x = qEt\,, \quad p_y = p_z = 0$

$$\Rightarrow \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = qEt \Rightarrow m^2\left(\frac{dx}{dt}\right)^2 = q^2E^2t^2\left(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{qEt}{\sqrt{m^2 + \frac{q^2}{c^2}E^2t^2}}$$

$$\Rightarrow x(t) = \int_0^t dt' \frac{qEt'}{\sqrt{m^2 + \frac{q^2}{c^2}E^2t'^2}} = \frac{q\cancel{E}}{\cancel{m}} \frac{c^2 m^{\cancel{2}}}{q^{\cancel{2}}E^2}\left(\sqrt{1+\frac{q^2E^2}{m^2c^2}t^2} - 1\right)$$

assume $X(0)=0$

$$\Rightarrow \boxed{x(t) = \frac{mc^2}{qE}\left(\sqrt{1+\frac{q^2E^2}{m^2c^2}t^2} - 1\right)}$$

moves with speed of light!

$\Rightarrow$ as $t \to \infty \Rightarrow x(t) \simeq c\, t.$ ~ linear in t!

$\Rightarrow$ if c is large $\Rightarrow$ expand in powers of $\frac{1}{c} \Rightarrow$

$$\Rightarrow x(t) \approx \frac{1}{2}\frac{qE}{m}t^2 = \frac{1}{2}at^2$$ ~ well-known classical NR result!

B. Constant Uniform Magnetic Field

$$\frac{d\vec{p}}{dt} = \frac{q}{c}\vec{v}\times\vec{B}, \quad \frac{dE}{dt} = 0 \Rightarrow E = \text{const.}$$

$$\Rightarrow \text{write } \vec{p} = m\gamma\vec{v} = m\gamma c^2 \cdot \frac{\vec{v}}{c^2} = E\cdot\frac{\vec{v}}{c^2}$$

$$\Rightarrow \frac{E}{c^2}\frac{d\vec{v}}{dt} = \frac{q}{c}\vec{v}\times\vec{B} \Rightarrow \text{define } \boxed{\vec{\omega}_B = \frac{q\vec{B}c}{E} = \frac{q\vec{B}}{\gamma mc}}$$

(precession frequency)

$$\Rightarrow \frac{d\vec{v}}{dt} = \vec{v}\times\vec{\omega}_B \quad \Rightarrow \text{if } \vec{B} = B\hat{z} \Rightarrow \vec{\omega}_B = \omega_B\hat{z}$$

$\Rightarrow$ get $\dot{V}_x = \omega_B V_y$, $\dot{V}_y = -\omega_B V_x$, $\dot{V}_z = 0$

$$\Rightarrow \ddot{V}_x = \omega_B \dot{V}_y = -\omega_B^2 V_x \Rightarrow V_x = V_{0\perp} \cdot e^{\pm i \omega_B t}$$

$$\Rightarrow V_y = \frac{1}{\omega_B} \dot{V}_x = \pm i V_{0\perp} e^{\pm i \omega_B t}$$

$\Rightarrow$ taking real parts write $V_x = V_{0\perp} \cos(\omega_B t + \alpha)$

$$\Rightarrow V_y = - V_{0\perp} \sin(\omega_B t + \alpha) \Rightarrow \sqrt{V_x^2 + V_y^2} = V_{0\perp}$$

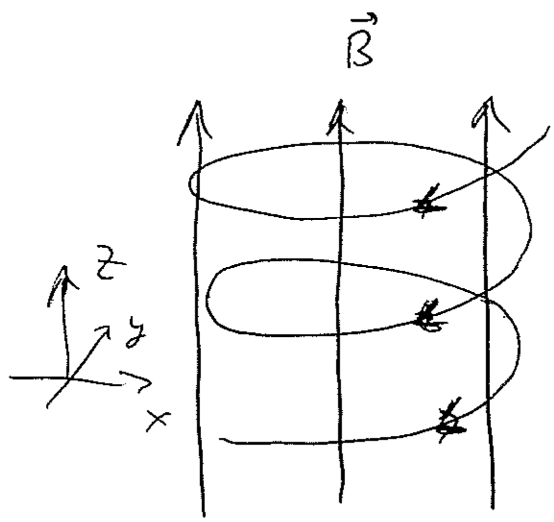
~ transverse (w.r.to $\vec{B}$) velocity

$\Rightarrow$ as $V_x = \dot{x} = V_{0\perp} \cos(\omega_B t + \alpha) \Rightarrow$

$$\Rightarrow \begin{cases} x(t) = x_0 + r \sin(\omega_B t + \alpha) \\ y(t) = y_0 + r \cos(\omega_B t + \alpha) \\ z(t) = z_0 + V_{0z} t \end{cases}$$

$$r = \frac{V_{0\perp}}{\omega_B} = \frac{V_{0\perp} E}{qBc} = \frac{c\, p_{0\perp}}{qB}.$$

Motion of a positive charge is shown here $\rightarrow$



can be shown to be conserved

Adiabatic Invariant: $\overset{\downarrow}{I} = \oint \vec{P}\cdot d\vec{l}$ ~ periodic motion.
(aka action integral) particle path

$\Rightarrow I = \oint \gamma m \vec{v}\cdot d\vec{l} + \frac{q}{c}\oint \vec{A}\cdot d\vec{l} \Rightarrow$

$\Rightarrow \oint \gamma m \vec{v}\cdot d\vec{l} = \gamma m \omega_B\, r \cdot 2\pi r = 2\pi r^2 \omega_B \gamma m$

$= 2\pi r^2 \overbrace{\frac{qB}{\cancel{\gamma m} c}}^{\omega_B} \cancel{\gamma m} = 2\pi \frac{q}{c} B r^2$

as $\oint \vec{A}\cdot d\vec{l} = \int da\, \hat{n}\cdot(\vec{\nabla}\times\vec{A})$ (Stokes's th'm)

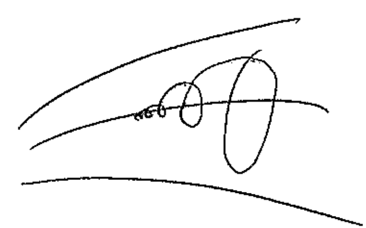
counterclockwise

$\frac{q}{c}\oint \vec{A}\cdot d\vec{l} = \frac{q}{c}\oint \vec{B}\cdot\hat{n}\, da \simeq -\frac{q}{c}\cdot B\pi r^2$

[Orbital angular momentum = $p\cdot r \propto Br^2$ "constant" $\Rightarrow Br^2 = \text{const.}$]

$\Rightarrow I \simeq \frac{q}{c}\pi r^2 B \Rightarrow$ the flux of B-field through the loop is $\pi r^2 \cdot B$ ~ it's invariant!

Example ~ particles moving in Earth's magnetic field ~ radius changes as B changes to keep flux constant.



Remember that $p_\perp = \frac{qBr}{c} \Rightarrow r = \frac{c p_\perp}{qB}$

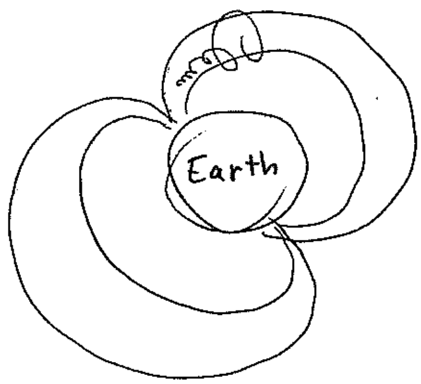
$\Rightarrow I = \frac{\cancel{q}}{\cancel{c}}\pi \frac{c^{\cancel{2}} p_T^2}{q^2 B^{\cancel{2}}}\cdot\cancel{B} = \frac{\pi c}{q}\frac{p_T^2}{B}$

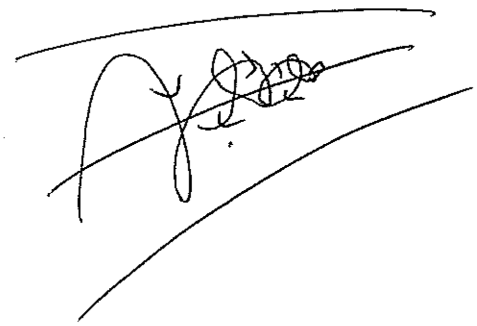
$\Rightarrow \frac{p_T^2}{B} = \text{const}$ $\Rightarrow$ as $E = c\sqrt{p_\perp^2 + p_z^2 + m^2c^2} \geq p_\perp c$

$\Rightarrow \frac{p_\perp'^2}{B'} = \frac{p_\perp^2}{B}$ ← initial, $E = \text{const} \Rightarrow p_\perp'^2 + p_z'^2 = p_\perp^2 + p_z^2 \Rightarrow$

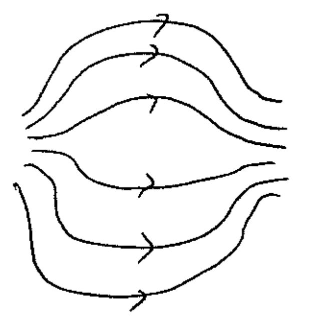
$$p_z'^2 = p_z^2 + p_\perp^2 - p_\perp'^2 = p_z^2 + \left(-\frac{B'}{B}+1\right) p_\perp^2 \geq 0$$

$\Rightarrow$ as $B' \gg B$ (particle enters strong magnetic field) $\Rightarrow$ eventually get $p_z' = 0$ $\Rightarrow$ the particle gets reflected back:

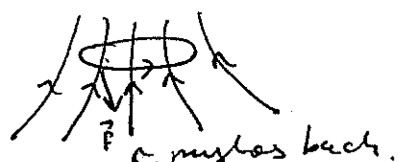




$\Rightarrow$ particle trapping in plasmas:

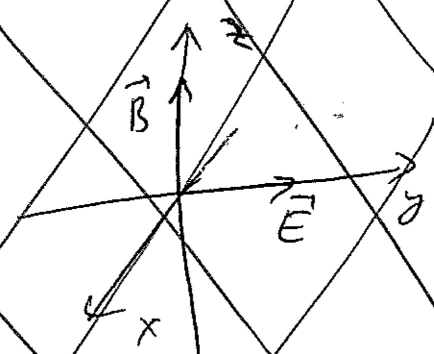


reflection



C. Constant Uniform Electric and Magnetic Fields.

$$\frac{d\vec p}{dt} = q\left(\vec E + \frac{1}{c}\vec v \times \vec B\right).$$



choose $\vec B = B\hat z$

~~and $\vec E$ in the plane yz~~

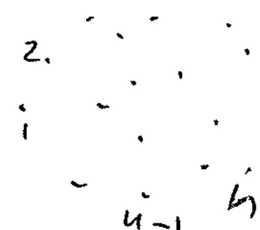
$\Rightarrow$ ~~a~~ choose $\vec E = E\hat y$. (for simplicity)

(here we consider $\vec E \perp \vec B$ only)

Lagrangian for Electromagnetic Field and Maxwell Equations

Four-Vector of Electromagnetic Current

Suppose we have N point charges:



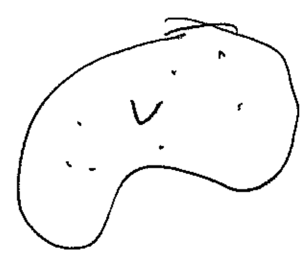
It is convenient to describe them in terms of charge density $\rho(\vec{x},t)$ (especially if n is large):

(Def.) Charge density $\rho(\vec{x},t)$ is defined as the electric charge per unit volume:

$$\boxed{\Delta q = \rho(\vec{x},t)\Delta x \Delta y \Delta z = \rho(\vec{x},t)\Delta V}$$

The net electric charge in volume V is then

$$\boxed{Q(t) = \int_V d^3x \ \rho(\vec{x},t)}$$



where $d^3x = dx\,dy\,dz$ is the volume integration measure