Last time | **Motion of a Point Charge in External $\vec{E}$ & $\vec{B}$ Fields** (cont'd)

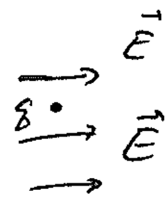
$$\begin{cases} \dfrac{d\vec{p}}{dt} = q\vec{E} + \dfrac{q}{c}\vec{v}\times B \\[2ex] \dfrac{d\mathcal{E}}{dt} = q\,\vec{v}\cdot\vec{E} \end{cases}$$

A. Constant Uniform $\vec{E}$-Field

$$\frac{d\vec{p}}{dt} = q\vec{E} \Rightarrow x(t) = \frac{mc^2}{qE}\left[\sqrt{1+\frac{q^2E^2}{m^2c^2}t^2} - 1\right]$$

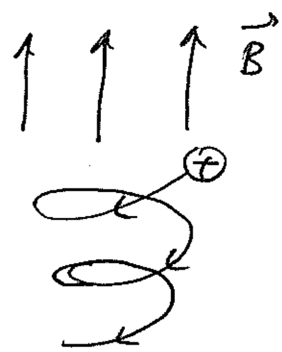
$\vec{E} = E\hat{x}$

charge starts from rest



B. Constant Uniform Magnetic Field

$$\frac{d\vec{p}}{dt} = \frac{q}{c}\vec{v}\times\vec{B}, \quad \frac{d\mathcal{E}}{dt} = 0, \quad \vec{B} = B\hat{z}$$

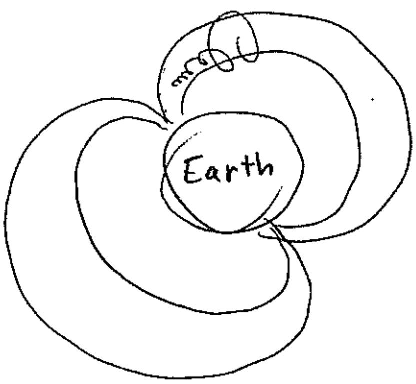


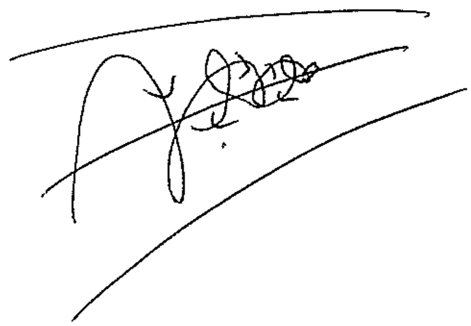
$$\begin{cases} x(t) = x_0 + r\sin(\omega_B t + \alpha) \\ y(t) = y_0 + r\cos(\omega_B t + \alpha) \\ z(t) = z_0 + V_{0z}t \end{cases}$$

where $r = \dfrac{V_{0\perp}}{\omega_B} = \dfrac{V_{0\perp}\mathcal{E}}{qBc} = \dfrac{c\,p_{0\perp}}{qB}$, $\quad \omega_B = \dfrac{qBc}{\mathcal{E}} = \dfrac{qB}{\gamma mc}$.

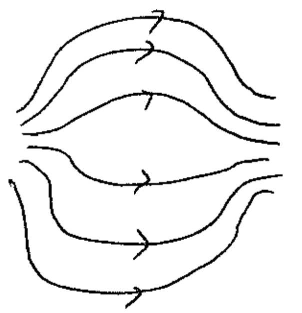
$$p_z'^2 = p_z^2 + p_\perp^2 - p_\perp'^2 = p_z^2 + \left(-\frac{B'}{B}+1\right) p_\perp^2 \geq 0$$

$\Rightarrow$ as $B' \gg B$ (particle enters strong magnetic field) $\Rightarrow$ eventually get $p_z' = 0$ $\Rightarrow$ the particle gets reflected back:

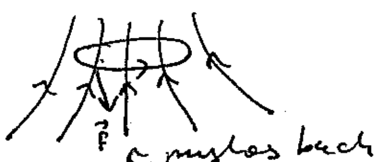




$\Rightarrow$ particle trapping in plasmas:

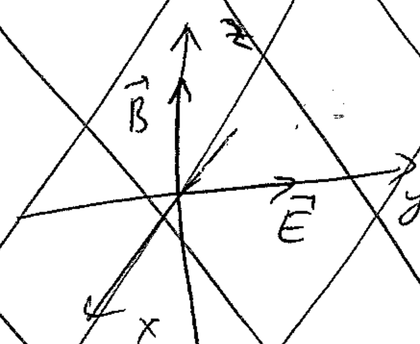


reflection



C. Constant Uniform Electric and Magnetic Fields.

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right).$$



choose $\vec{B} = B\hat{z}$

~~and $\vec{E}$ in the plane yz~~

$\Rightarrow$ choose $\vec{E} = E\hat{y}$. (for simplicity)

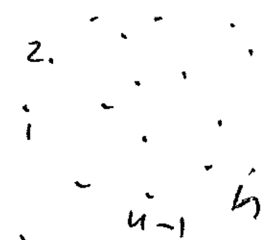
(here we consider $\vec{E} \perp \vec{B}$ only)

Lagrangian for Electromagnetic Field and Maxwell Equations

Four-Vector of Electromagnetic Current

Suppose we have N point charges:

It is convenient to describe them in terms of charge density $\rho(\vec{x},t)$ (especially if n is large):

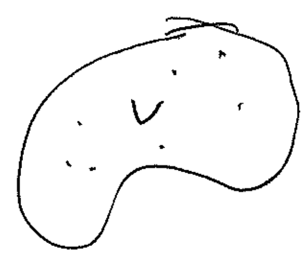


(Def.) Charge density $\rho(\vec{x},t)$ is defined as the electric charge per unit volume:

$$\boxed{\Delta q = \rho(\vec{x},t)\Delta x\,\Delta y\,\Delta z = \rho(\vec{x},t)\Delta V}$$

The net electric charge in volume V is then

$$\boxed{Q(t) = \int_V d^3x \;\rho(\vec{x},t)}$$



where $d^3x = dx\,dy\,dz$ is the volume integration measure

- For a discrete set of charges $q_1, \dots, q_n$ at locations $\vec{x}_1, \dots, \vec{x}_n$ we write

$$\boxed{\rho(\vec{x},t) = \sum_{i=1}^{n} q_i \, \delta^3(\vec{x} - \vec{x}_i(t))}.$$

where $\delta^3(\vec{x})$ is the Dirac <u>delta</u>-<u>function</u>.

Definition of Dirac delta function:

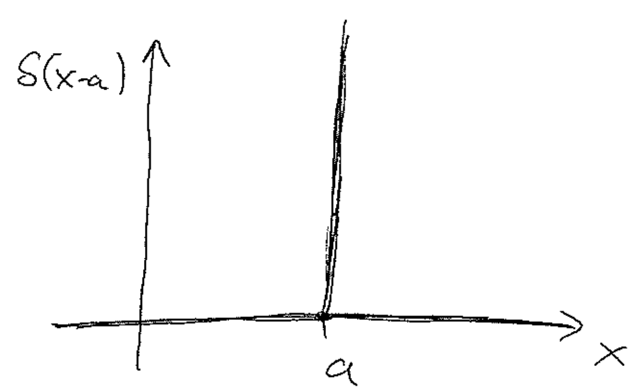
(i) $$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \text{ (that is, ill-defined at } x=a) \end{cases}$$

(ii) $$\int_{-\infty}^{\infty} dx \, f(x)\, \delta(x-a) = f(a)$$ ~ real definition not (i)

for $f(x)$ which is infinitely differentiable ("well-behaved" function)

In particular, for $f(x) = 1$ we get

$$\int_{-\infty}^{\infty} dx \, \delta(x-a) = 1.$$



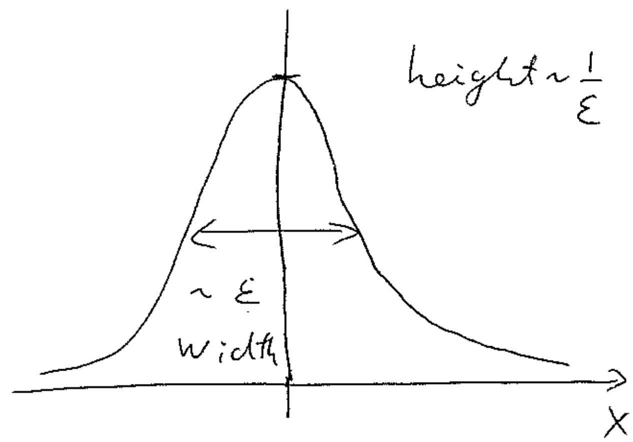
(see Artken Section 1.11, Jackson page 26)

δ-fn is a functional (distribution)

Delta function can be thought of as a limit of some **bell-shaped** curve:

let's show that $\delta(x) = \lim_{\varepsilon \to 0} \left\{ \frac{1}{\sqrt{\pi}\,\varepsilon} e^{-x^2/\varepsilon^2} \right\}$

check (i) and (ii):



(i) if $x \neq 0 \Rightarrow$

$$\lim_{\varepsilon \to 0} \left\{ \frac{1}{\sqrt{\pi}\,\varepsilon} e^{-x^2/\varepsilon^2} \right\} = \emptyset.$$

if $x = 0$

$$\lim_{\varepsilon \to 0} \left\{ \frac{1}{\sqrt{\pi}\,\varepsilon} \right\} = \infty.$$

(ii) to simplify our life, let's work only with $f(x) = 1$:

$$\int_{-\infty}^{\infty} dx \, \frac{1}{\sqrt{\pi}\,\varepsilon} e^{-x^2/\varepsilon^2} = \left| \begin{array}{l} \text{define} \\ \xi \equiv \frac{x}{\varepsilon} \end{array} \right. = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \, e^{-\xi^2} = 1$$

for $\forall\, \varepsilon \Rightarrow$ (ii) is satisfied.

(to do $\int_{-\infty}^{\infty} d\xi \, e^{-\xi^2} = I$ let's find $I^2 = \int_{-\infty}^{\infty} dx\,dy\, e^{-x^2-y^2} = \left| \begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array} \right.$

$= \int_0^{\infty} dr \cdot r\, e^{-r^2} \cdot \int_0^{2\pi} d\varphi = 2\pi \cdot \left(-\frac{1}{2} e^{-r^2} \right) \Big|_0^{\infty} = \pi$)

to prove (ii) for a more general class of functions go to Fourier transform

$$f(x) = \int \frac{dk}{2\pi} e^{-ik\cdot x} \tilde{f}(k)$$

& will have to prove (ii) only for exponents

$$f(x) \sim e^{-ik\cdot x}$$]

Properties of delta-functions:

(1) $\delta(-x) = \delta(x)$ (it's an even function)

(2.) $$\int_{-\infty}^{\infty} dx\, f(x)\, \delta^{(n)}(x-a) = (-1)^n f^{(n)}(a)$$

in particular $$\int_{-\infty}^{\infty} dx\, f(x)\, \delta'(x-a) = -f'(a)$$

Proof: $$\int_{-\infty}^{\infty} dx\, f(x) \frac{d}{dx} \delta(x-a) \overset{\text{parts}}{=} f(x)\,\delta(x-a)\Big|_{-\infty}^{\infty} -$$

$$- \int_{-\infty}^{\infty} dx\, f'(x)\, \delta(x-a) = -f'(a)$$

(3) $$\delta(f(x)) = \sum_{i=1}^{n} \frac{1}{|f'(x_i)|} \delta(x - x_i)$$

where $x = 1, \ldots, n$ are roots of $f(x)$, $f(x_i) = \emptyset$.

Proof: $$1 = \int d f(x)\; \delta(f(x)) = \int_{x_i-\Delta}^{x_i+\Delta} dx \cdot |f'(x)|\, \delta(f(x))$$

integrate near one of the roots x_i

(need abs value $|f'(x)|$ to have the right direction of the integral over x, $\int_{x_i-\Delta}^{x_i+\Delta}$ and not $\int_{x_i+\Delta}^{x_i-\Delta}$).

$\Rightarrow$ we see that $$\delta(x-x_i) = |f'(x)|\, \delta(f(x))$$

for x near x_i $\Rightarrow$ $$\delta(f(x)) = \frac{1}{|f'(x_i)|}\, \delta(x-x_i)$$

in the vicinity of x_i $\Rightarrow$ $$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\, \delta(x-x_i)$$

after summing over all roots.

(4) $$\delta^3(\vec{x}-\vec{y}) = \delta(x_1-y_1)\, \delta(x_2-y_2)\, \delta(x_3-y_3)$$

(can treat this as a definition of δ-ftn.) in 3d.

~~Gauss's Law~~

~~Consider a point charge q inside some closed surface S.~~