Last time

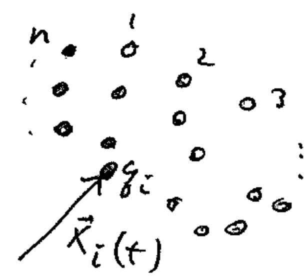
Lagrangian for the Electromagnetic Field and Maxwell Equations (cont'd)

Four-Vector of Electromagnetic Current (cont'd)

Def. charge density $\boxed{\rho(\vec{x},t) = \frac{\text{charge}}{\text{Volume}}}$

For a discrete set of point charges q_i, $i=1,\dots,n$

get $\boxed{\rho(\vec{x},t) = \sum_{i=1}^{n} q_i\, \delta^3(\vec{x} - \vec{x}_i(t))}$



Def. Dirac delta-function:

(i) $\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases}$

(ii) $\int_{-\infty}^{\infty} dx\, f(x)\, \delta(x-a) = f(a)$.

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi}\,\varepsilon} e^{-x^2/\varepsilon^2}$$

~ can be thought of as a limit of a smooth function

Properties of δ-function:

(1) $\delta(x) = \delta(-x)$

(2) $\int_{-\infty}^{\infty} dx\, f(x)\, \delta^{(n)}(x-a) = (-1)^n f^{(n)}(a)$

(3) $\delta(f(x)) = \sum_{i=1}^{n} \frac{1}{|f'(x_i)|} \delta(x - x_i)$, x_i with $i = 1, \dots, n$

are roots of $f(x)$, such that $f(x_i) = 0$.

(4) $\delta^3(\vec{x} - \vec{y}) = \delta(x^1 - y^1)\, \delta(x^2 - y^2)\, \delta(x^3 - y^3)$

also

$\delta^4(x - y) = \delta(x^0 - y^0)\, \delta(x^1 - y^1)\, \delta(x^2 - y^2)\, \delta(x^3 - y^3)$

4-dimensional δ-function, x^μ & y^μ are 4-vectors

To prove (ii) for a more general class of functions go to Fourier transform

$$f(x) = \int \frac{dk}{2\pi} e^{-ik\cdot x} \tilde{f}(k)$$

& will have to prove (ii) only for exponents

$f(x) \sim e^{-ik\cdot x}$]

Properties of delta-functions:

(1) $\delta(-x) = \delta(x)$ (it's an even function)

(2.) $$\int_{-\infty}^{\infty} dx\, f(x)\, \delta^{(n)}(x-a) = (-1)^n f^{(n)}(a)$$

in particular $\int_{-\infty}^{\infty} dx\, f(x)\, \delta'(x-a) = -f'(a)$

Proof: $$\int_{-\infty}^{\infty} dx\, f(x) \frac{d}{dx}\delta(x-a) \overset{\text{parts}}{=} f(x)\,\delta(x-a)\Big|_{-\infty}^{\infty} -$$

$$- \int_{-\infty}^{\infty} dx\, f'(x)\, \delta(x-a) = -f'(a)$$

(3) $$\delta(f(x)) = \sum_{i=1}^{n} \frac{1}{|f'(x_i)|} \delta(x - x_i)$$

where x_i, $i = 1, \ldots, n$ are roots of $f(x)$, $f(x_i) = \emptyset$.

Proof: $$1 = \int d f(x)\ \delta(f(x)) = \int_{x_i-\Delta}^{x_i+\Delta} dx \cdot |f'(x)|\, \delta(f(x))$$

integrate near one of the roots x_i

(need abs value $|f'(x)|$ to have the right direction of the integral over x, $\int_{x_i-\Delta}^{x_i+\Delta}$ and not $\int_{x_i+\Delta}^{x_i-\Delta}$).

$\Rightarrow$ we see that $\delta(x-x_i) = |f'(x)|\, \delta(f(x))$

for x near x_i $\Rightarrow$ $\delta(f(x)) = \dfrac{1}{|f'(x_i)|}\delta(x-x_i)$

in the vicinity of x_i $\Rightarrow$ summing over all roots $\delta(f(x)) = \sum_i \dfrac{1}{|f'(x_i)|}\delta(x-x_i)$

after summing over all roots.

(4) $\delta^3(\vec{x}-\vec{y}) = \delta(x_1-y_1)\,\delta(x_2-y_2)\,\delta(x_3-y_3)$

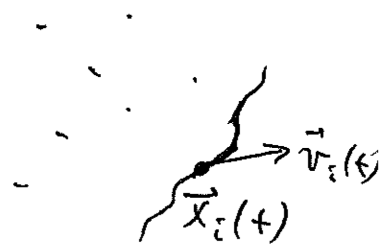
(can treat this as a definition of δ-ftn.) on 3d.

Also $\delta^4(x-y) = \delta(x^0-y^0)\,\delta(x^1-y^1)\,\delta(x^2-y^2)\,\delta(x^3-y^3)$

4-dim δ-function, x^μ & y^μ are 4-vectors

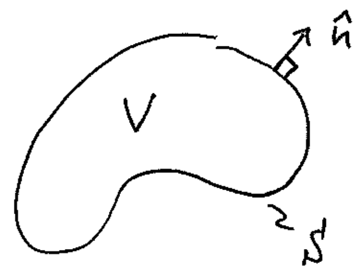
Def. Current density $\vec{J}(\vec{x},t)$ is defined as the current per unit area or

$$J = \frac{\text{charge} \cdot \text{velocity}}{\text{volume}}$$



For point charges write

$$\boxed{\vec{J}(\vec{x},t) = \sum_{i=1}^{n} q_i \, \vec{v}_i(t) \, \delta^3(\vec{x} - \vec{x}_i(t))}$$



Charge conservation:

Imagine a volume V: the change in total charge inside the volume is equal to the amount of charge that flowed in/out the volume:

$$\Delta Q = \int_V d^3x \left[\rho(\vec{x}, t+\Delta t) - \rho(\vec{x},t) \right] = -\Delta t \oint_S da \; \hat{n} \cdot \vec{J}$$

where $\hat{n}$ is a unit normal to the surface vector pointing outward, da ~ surface element

Divergence Theorem for a vector field $\vec{V}(\vec{x})$ we have (see Arfken Sec. 3.8)

$$\boxed{\int_V d^3x \; \vec{\nabla} \cdot \vec{V} = \oint_S da \; \hat{n} \cdot \vec{V}}$$

Using the divergence theorem we write

$$\oint_S da\, \hat{n}\cdot\vec{J} = \int_V d^3x\, \vec{\nabla}\cdot\vec{J}$$

such that

$$\int_V d^3x \left[\rho(\vec{x},t+\Delta t)-\rho(\vec{x},t)\right] = -\int_V d^3x\, \vec{\nabla}\cdot\vec{J}\cdot\Delta t$$

=> since the volume is chosen arbitrarily, equate the integrands =>

$$\rho(\vec{x},t+\Delta t)-\rho(\vec{x},t) \approx \Delta t\,\frac{\partial\rho(\vec{x},t)}{\partial t} = -\Delta t\,\vec{\nabla}\cdot\vec{J}$$

$$\Rightarrow \quad \boxed{\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{J}=0}$$

continuity equation ~ manifests charge conservation

As we know $x^\mu = (ct, \vec{x})$ and $\partial_\mu = \frac{\partial}{\partial x^\mu}$ are 4-vectors => defining an object

$$\boxed{J^\mu = (c\rho, \vec{J})}$$

we rewrite the continuity equation as

$$\boxed{\partial_\mu J^\mu = 0}.$$

This is true in any frame => a Lorentz-invariant statement. As $\partial_\mu J^\mu$ is Lorentz-invariant, and ∂_μ is a 4-vector => J^μ is a 4-vector too!

$\Rightarrow \boxed{J^\mu}$ is a 4-vector of current

$\boxed{\partial_\mu J^\mu = 0}$ is often referred to as the current conservation

The action for charge-field interactions is

$S_{int} = -\frac{q}{c}\int dt \frac{1}{\gamma} u_\mu A^\mu \Rightarrow$ for the set of n point charges at hand write

$$S_{int} = -\frac{1}{c}\int dt\, d^3x \underbrace{\sum_i q_i \frac{1}{\gamma_i} u_\mu^i \delta^3(\vec{x}-\vec{x}_i)}_{J_\mu} A^\mu(x)$$

Since
$$\begin{cases} c\rho = c\sum_i q_i \delta^3(\vec{x}-\vec{x}_i) \\ \vec{J} = \sum_i q_i \vec{v}_i \delta^3(\vec{x}-\vec{x}_i) \end{cases}$$
& as $u^\mu = (c, \vec{v})\gamma$

$$\Rightarrow \quad J_\mu = (c\rho, \vec{J}) = \sum_i q_i \frac{1}{\gamma} u_\mu^i \delta^3(\vec{x}-\vec{x}_i)$$

$$\Rightarrow \quad S_{int} = -\frac{1}{c}\int dt\, d^3x\, J_\mu A^\mu$$

Define a 4-dim integration measure

$$\boxed{d^4x \equiv dx^0 dx^1 dx^2 dx^3 = c\, dt\, d^3x}$$

Note that d^4x is Lorentz-invariant:

under Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

we get $d^4x' = |\det \Lambda|\, d^4x$, but $\det \Lambda = +1$

$\Rightarrow d^4x' = d^4x$ (Lorentz-invariant)

e.g.
$$\det \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \gamma^2(1-\beta^2) = 1 \quad \text{(true for all } \Lambda)$$

$\Rightarrow$ Using d^4x write

$$\boxed{S_{int} = -\frac{1}{c^2}\int d^4x\, J_\mu A^\mu}$$

In general, $S_{int} = \int d^4x\, \mathcal{L}_{int}$, where $\mathcal{L}$ is called the Lagrangian density (such that $L = \int d^3x\, \mathcal{L}$).

We get
$$\mathcal{L}_{int} = -\frac{1}{c^2} J_\mu A^\mu$$

Note that
$$\underbrace{S}_{\text{L. inv.}} = \int \underbrace{d^4x}_{\text{L. inv.}}\ \underbrace{\mathcal{L}}_{\text{L. inv.}}$$
$\Rightarrow \mathcal{L}$ is Lorentz-invariant in general.