Last time | Def. Current density $\vec{J}(\vec{x},t)$:

$$J = \frac{\text{charge} \cdot \text{velocity}}{\text{volume}}$$

$$\boxed{\vec{J}(\vec{x},t) = \sum_{i=1}^{n} q_i \, \vec{v}_i(t) \, \delta^3(\vec{x} - \vec{x}_i(t))}$$

Charge conservation $\Rightarrow$ $\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$

continuity equation

$\boxed{J^\mu = (c\rho, \vec{J})}$ ~ 4-vector of current

The action for the matter-field interactions can be written as

$$\boxed{S_{int} = -\frac{1}{c^2} \int d^4x \, J_\mu A^\mu}$$

$d^4x$ ~ Lorentz-invariant

$S_{int}$ ~ — " —

$\mathcal{L}_{int} = -\frac{1}{c^2} J_\mu A^\mu$ ~ Lagrangian density $\Rightarrow$ also Lorentz-invariant.

# Field Strength Tensor Revisited and Transformation of $\vec{E}$ and $\vec{B}$ under Boosts

$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$ ~ field-strength tensor

$F_{\mu\nu} = -F_{\nu\mu}$ ~ anti-symmetric tensor, rank-2

$\boxed{F^{0i} = -E^i}$, $\boxed{F^{ij} = -\varepsilon^{ijk} B^k}$ => components of $F^{\mu\nu}$ are $\vec{E}$ & $\vec{B}$ fields.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Def. Define a <u>dual field-strength tensor</u>:

$$\boxed{\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}}$$

where $\varepsilon^{0123} = 1$, $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma}$ ~ changes sign under permutations and $\varepsilon^{\mu\alpha\rho\alpha} = \varepsilon^{\alpha\alpha\rho\sigma} = 0$ (no 2 indices are the same) ~ 4-dim Levi-Civita symbol (rank-4 tensor).

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

duality trans

$\vec{E} \to \vec{B}$

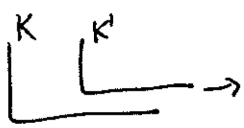
$\vec{B} \to -\vec{E}$.

Let us find the transformation laws of $\vec{E}$ & $\vec{B}$ fields under Lorentz boosts.

Note that $\vec{E}$ & $\vec{B}$ are not parts of some 4-vectors, but are components of a rank-2 tensor $F^{\mu\nu}$.

$\Rightarrow$ Let's Lorentz-transform $F_{\mu\nu}$ and see what happens to its components:

$$F'^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu}\, \Lambda^{\nu'}{}_{\nu}\, F^{\mu\nu} = \Lambda^{\mu'}{}_{\mu}\, F^{\mu\nu}\, \Lambda_{\nu}{}^{\nu'}$$




as $\Lambda^{\nu'}{}_{\nu} = \Lambda_{\nu}{}^{\nu'}$ or $\Lambda = \Lambda^T$ (for boosts only)

$$\Rightarrow F'^{\mu'\nu'} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \cdot$$

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & (\beta^2\gamma^2-\gamma^2)E_x & -\gamma E_y+\beta\gamma B_z & -\gamma E_z-\beta\gamma B_y \\ (\gamma^2-\beta^2\gamma^2)E_x & 0 & \beta\gamma E_y-\gamma B_z & \beta\gamma E_z+\gamma B_y \\ \gamma E_y-\beta\gamma B_z & -\beta\gamma E_y+\gamma B_z & 0 & -B_x \\ \gamma E_z+\beta\gamma B_y & -\beta\gamma E_z-\gamma B_y & B_x & 0 \end{pmatrix}$$

$$\Rightarrow \text{as } F'^{\mu'\nu'} = \begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x & 0 & -B_z' & B_y' \\ E_y & B_z' & 0 & -B_x' \\ E_z & -B_y' & B_x' & 0 \end{pmatrix} \Rightarrow$$

$$\boxed{\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - \beta B_z) & B_y' &= \gamma(B_y + \beta E_z) \\ E_z' &= \gamma(E_z + \beta B_y) & B_z' &= \gamma(B_z - \beta E_y) \end{aligned}}$$

if $v \ll c \Rightarrow$ get $\vec{E}' = \vec{E} + \frac{\vec{v}}{c} \times \vec{B}$

$\vec{B}' = \vec{B} - \frac{1}{c}\vec{v} \times \vec{E}$. (Galilean transformations for $\vec{E}$ & $\vec{B}$)

<u>Lorentz - invariants:</u>

$$\boxed{F^{\mu\nu}F_{\mu\nu} = 2\left(\vec{B}^2 - \vec{E}^2\right)} = -\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}$$

~ by construction this is Lorentz-inv.

$$\boxed{F^{\mu\nu}\tilde{F}_{\mu\nu} = -4\,\vec{B}\cdot\vec{E}}$$

~ also Lorentz-inv.

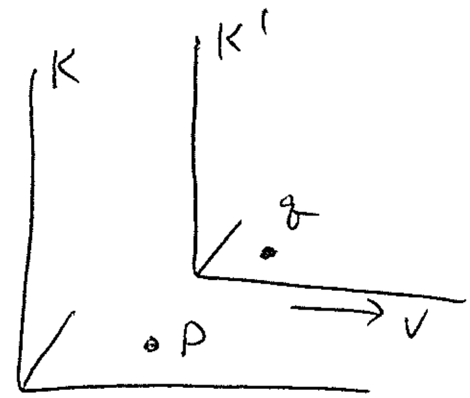
<u>Example</u>: plane waves, $\vec{E} = -\frac{c}{\omega}\vec{k} \times \vec{B} \Rightarrow$

$\Rightarrow \vec{E}\cdot\vec{B} = 0$ , $|\vec{E}| = |\vec{B}| \Rightarrow E^2 - B^2 = 0$

~ true in all frames!

<u>Example</u>: moving point charge: in it's rest frame the field is given by Coulomb's law: $\vec{E}' = \frac{q}{r'^3}\vec{r}'$

(note Gaussian units)



$\Rightarrow$ observer at $P$ has coordinates $(x_1', x_2', x_3')$

$\Rightarrow$ $\vec{E}'(P) = q \dfrac{\vec{x}'}{\vec{x}'^3}$ (assume that charge $q$ is at the origin in frame $K'$)

$\vec{B}' = 0$

$\Rightarrow$ boost $\to$ $E_x = E_x' = q \dfrac{x'}{(x'^2 + y'^2 + z'^2)^{3/2}} =$

$$= q \frac{\gamma (x - vt)}{[\gamma^2 (x - vt)^2 + y^2 + z^2]^{3/2}}$$

$$E_y = \gamma (E_y' + \beta \underbrace{B_z'}_{0}) = q \frac{\gamma y}{[\gamma^2 (x - vt)^2 + y^2 + z^2]^{3/2}}$$

$$E_z = \gamma (E_z' - \beta \underbrace{B_y'}_{=0}) = q \frac{\gamma z}{[\gamma^2 (x - vt)^2 + y^2 + z^2]^{3/2}}$$

$$B_x = B_x' = 0 \, ; \quad B_y = \gamma (\overset{=0}{B_z'} - \beta E_z') =$$

$$= -\gamma \beta \frac{q z}{[\gamma^2 (x - vt)^2 + y^2 + z^2]^{3/2}}$$

$$B_z = \gamma (\underbrace{B_z'}_{=0} + \beta E_y') = \gamma \beta \frac{q y}{[\gamma^2 (x - vt)^2 + y^2 + z^2]^{3/2}}$$

In the non-relativistic case: $\beta \ll 1$, $\gamma \simeq 1$

$$\Rightarrow B_y \simeq -\frac{v}{c} \frac{q z}{[(x - vt)^2 + y^2 + z^2]^{3/2}} \, ; \quad B_z \simeq \frac{v}{c} \frac{q y}{[(x - vt)^2 + y^2 + z^2]^{3/2}}$$