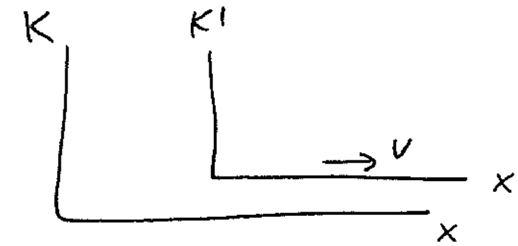
Last time | Field Strength Tensor Revisited: Transformation of $\vec{E}$ & $\vec{B}$ under boosts (cont'd)

used the fact that $F^{\mu\nu}$ is a Lorentz tensor to infer the following:



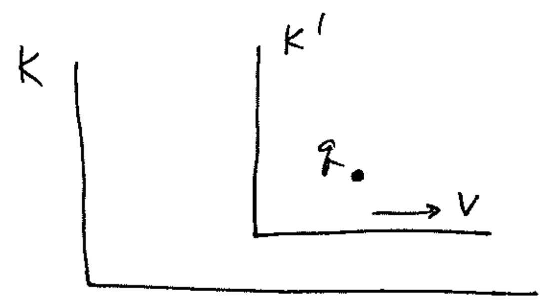
$$\boxed{\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - \beta B_z) & B_y' &= \gamma(B_y + \beta E_z) \\ E_z' &= \gamma(E_z + \beta B_y) & B_z' &= \gamma(B_z - \beta E_y) \end{aligned}}$$

Lorentz invariants: $F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = -4\, \vec{B} \cdot \vec{E}$$

Example | Point charge q moving with velocity v:

Using $\vec{E}$ & $\vec{B}$ field transformations from above we found:



$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{q\gamma}{\left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{3/2}} \begin{pmatrix} x - vt \\ y \\ z \end{pmatrix}$$

along with

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{q \gamma \beta}{\left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{3/2}} \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}.$$

$$\Rightarrow \vec{B} = \frac{q}{c}\,\frac{\vec{v}\times\vec{r}}{r^3} \Rightarrow \text{Biot - Savart Law!}$$

(as expected!)

$$\lim_{\gamma\to\infty} \frac{\gamma}{\big(\gamma^2\underbrace{(x-ct)^2}_{\xi}+x_\perp^2\big)^{3/2}} = \begin{cases} \ldots \\ \infty, & x = ct \end{cases} = \frac{\ }{x_\perp^2}\,\delta(x-ct)$$

$$x_\perp^2 = y^2+z^2$$

$$\int_{-\infty}^{\infty} d\xi\, \frac{\gamma}{\left[\gamma^2\xi^2+x_\perp^2\right]^{3/2}} = \Big| \tilde{\xi} = \gamma\xi \Big| = \int_{-\infty}^{\infty} \frac{d\tilde{\xi}}{\left[\tilde{\xi}^2+x_\perp^2\right]^{3/2}} = \frac{2}{x_\perp^2}$$

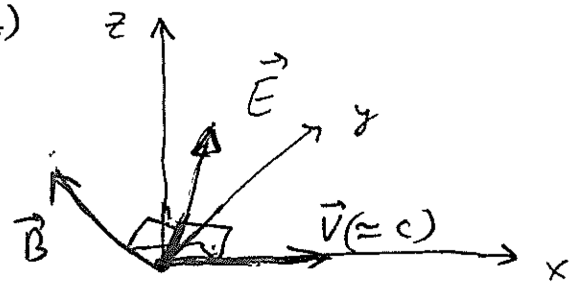
$$\Rightarrow \boxed{\begin{aligned} E_y &= q\,y\,\frac{2}{y^2+z^2}\,\delta(x-ct) \\ E_z &= q\,z\,\frac{2}{y^2+z^2}\,\delta(x-ct) \\ E_x &= 0 \qquad B_x = 0 \\ B_y &= -q\,z\,\frac{2}{y^2+z^2}\,\delta(x-ct) \\ B_z &= q\,y\,\frac{2}{y^2+z^2}\,\delta(x-ct) \end{aligned}}$$

as $\gamma \to \infty$

$\beta \to 1$

$$\underline{E} = 2q\,\frac{\underline{x}}{\underline{x}^2}\,\delta(x-ct), \quad \underline{x} \equiv (y, z)$$

$$\vec{B} = \hat{x} \times \vec{E} \quad \left(= \frac{c}{\omega}\,\vec{k}\times\vec{E} \right) \quad \text{for } \vec{k} = \frac{\omega}{c}\hat{x}.$$



→ looks like a plane wave frozen around the particle. ⇒ equivalent photon approximation

(same story for quarks & gluons in a proton)

Gauge Invariance

The 4-vector potential A^μ is not defined uniquely. Different $A^\mu(x)$ may give the same $\vec{E}$ & $\vec{B}$ $\Rightarrow$ $\Rightarrow$ the same $F^{\mu\nu}$.

Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, it is invariant under gauge transformations

$$A_\mu \to A_\mu' = A_\mu - \partial_\mu \Lambda(x)$$

where $\Lambda(x)$ is an arbitrary scalar function of x^μ.

Really, $F_{\mu\nu} \to F'_{\mu\nu} = \partial_\mu A_\nu' - \partial_\nu A_\mu' =$

$$= \partial_\mu A_\nu - \partial_\mu \partial_\nu \Lambda - \partial_\nu A_\mu + \partial_\nu \partial_\mu \Lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}.$$

$\Rightarrow$ Physics is invariant under gauge transforms

$\Rightarrow$ gauge-invariance! $\Rightarrow$ gauge theories are at the core of our understanding of Nature $\Rightarrow$ E&M is the first such theory we encountered

In terms of components: (as $\nabla^i \equiv \partial_i$)

$$\boxed{\begin{aligned} \Phi &\to \Phi' = \Phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} \\ \vec{A} &\to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \end{aligned}}$$

gauge transformations

One can see that $\vec{E}$ & $\vec{B}$ fields are invariant under gauge transformations explicitly:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \to \vec{E}' = -\vec{\nabla}\Phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\left(\Phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right)$$

$$-\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\Lambda\right) = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{E}.$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \to \vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\left(\vec{A} + \vec{\nabla}\Lambda\right) = \vec{\nabla}\times\vec{A} = \vec{B}$$

since $\vec{\nabla}\times\vec{\nabla}\Lambda = 0$ $\quad \Big(\underbrace{\varepsilon^{ijk}}_{\text{anti-symm}} \underbrace{\partial_j \partial_k}_{\text{symm.}} \Lambda = 0 \Big)$.

If the physics is gauge-invariant, the action should be as well. What about $S_{int} = -\frac{1}{c^2}\int d^4x \, J_\mu A^\mu$? Seems to depend on A^μ...

$$S_{int} = -\frac{1}{c^2}\int d^4x \, J_\mu A^\mu \to S'_{int} = -\frac{1}{c^2}\int d^4x \, J_\mu\left(A^\mu - \partial^\mu \Lambda\right)$$

$$= S_{int} + \frac{1}{c^2}\int d^4x \Big[\underbrace{\partial^\mu\left(J_\mu \Lambda\right)}_{\substack{\text{divergence theorem} \\ \Downarrow \\ \text{surface term} \Rightarrow \text{drop}}} - \Lambda \underbrace{\partial^\mu J_\mu}_{\substack{=0 \text{ (charge} \\ \text{conservation)}}}\Big] = S_{int}$$

$\Rightarrow$ S_{int} is gauge-invariant!

$\Rightarrow$ Gauge invariance is related to charge/current conservation in E&M.

Lagrangian for the Electromagnetic Field.

First let's discuss the differences between Lagrangians for fields vs. point particles:

for point particles $L = L(q_i, \dot{q}_i, t)$

and the action is $S = \int dt \, L(q_i, \dot{q}_i, t)$

$q_i \sim$ degrees of freedom (e.g. coordinates)

$\dot{q}_i = \dfrac{dq_i}{dt} \sim$ generalized velocities.

Suppose instead of discrete charges we'll have a field $\phi(\vec{x}, t)$ (e.g. wave-function for a particle in QM, or EM potential ...)

Classical Mechanics		Classical Field Theory
q_i	$\longrightarrow$	$\phi(\vec{x}, t)$
i	$\longrightarrow$	$\vec{x}, t$
$\dot{q}_i$	$\longrightarrow$	$\partial_\mu \phi(\vec{x}, t)$ $\mu = 0, 1, 2, 3$

$$L(q_i, \dot{q}_i) \rightarrow \int d^3x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

Lagrangian density

Such that the action is

$$S = \int dt \, L = \int \underbrace{dt\, d^3x}_{\frac{1}{c} d^4x \,\leftarrow\, \text{Lorentz scalar}} \mathcal{L}(\phi, \partial_\mu \phi) =$$

as $d^4x' = \left| \det \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| d^4x = d^4x$

$$= \frac{1}{c} \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

$\Rightarrow$ $\mathcal{L}$ is a Lorentz-scalar. (why?)

$$\Rightarrow \boxed{S = \frac{1}{c} \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)}$$

Let's find the equations of motion: have to vary the action S w.r.t. $\phi \rightarrow \phi + \Delta\phi$

$$\Rightarrow 0 \overset{\text{least action principle}}{=} \Delta S = \frac{1}{c} \int d^4x \left[\frac{\delta \mathcal{L}}{\delta \phi} \Delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \Delta(\partial_\mu \phi) \right]$$

$\Rightarrow$ as $\Delta(\partial_\mu \phi) = \partial_\mu(\Delta \phi)$ $\Rightarrow$ parts

$$0 = \frac{1}{c} \int d^4x \left[\frac{\delta \mathcal{L}}{\delta \phi} \Delta\phi - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \right) \Delta\phi \right] +$$

$+$ surface term $= 0$. $\Rightarrow$ this is true for any $\Delta\varphi$ $\Rightarrow$

require $\Delta\varphi = 0$ at $x^\mu \to \infty$. $\Rightarrow$ the integrand is zero

$$\Rightarrow \boxed{\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right) = 0}$$ Euler-Lagrange equations for a field ϕ.

Now, let's find $\mathcal{L}$ for EM fields, $\mathcal{L} = \mathcal{L}(A_\mu, \partial_\mu A_\nu)$

$\Rightarrow$ EM field have superposition principle ~ equations of motion (Maxwell equ's) are linear $\Rightarrow$ $\mathcal{L}$ has to be quadratic in A_μ.

$\Rightarrow$ $\mathcal{L}$ is a Lorentz-scalar and is gauge-invariant ($A_\mu \to A_\mu + \partial_\mu \Lambda$) $\Rightarrow$ the only quadratic invariants we can build out of $F^{\mu\nu}$ & $\tilde{F}^{\mu\nu}$ are

$I_1 \propto F_{\mu\nu} F^{\mu\nu}$ $(= -\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu})$ and $I_2 \propto F_{\mu\nu}\tilde{F}^{\mu\nu}$.

But: I_2 is a pseudo-scalar under parity ($I_2 \to -I_2$ if $\vec{x} \to -\vec{x}$) $\Rightarrow$ can't be in $\mathcal{L}$

(actually, I_2 can be written as $\partial_\mu K^\mu$, with k_μ some 4-vector $\Rightarrow \int d^4x\, I_2 = \int d^4x\, \partial_\mu K^\mu = \int_{\text{Surface}} d\sigma_\mu K^\mu = 0$)

$\Rightarrow$ $\mathcal{L} \propto F_{\mu\nu}F^{\mu\nu}$ $\Rightarrow$ picking normalization to get Maxwell equs (in Gaussian units) write $$\boxed{\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}}$$

Remember the interaction action

$$S_{int} = -\frac{q}{c}\int dt\,\frac{1}{\gamma}\,u_\mu A^\mu = -\frac{1}{c}\int dt\,d^3x \sum_i q_i \frac{1}{\gamma_i}\cdot$$

$$\cdot\, u^i_\mu \quad \delta^3(\vec{x}-\vec{x}_i)\,A^\mu(x)$$ for a set of discrete

charges $\{q_i\}$; $\left.\begin{array}{l} c\sum_i q_i\,\delta^3(\vec{x}-\vec{x}_i) \to c\rho(\vec{x}) \\ \sum_i q_i\,\vec{v}^i\,\delta^3(\vec{x}-\vec{x}_i) \to \vec{J}(\vec{x}) \end{array}\right\} J^\mu.$

As $\frac{u^i_\mu}{\gamma_i} = (c, \vec{v}^i) \Rightarrow S_{int} = -\frac{1}{c^2}\int d^4x\; J_\mu A^\mu$

where $J^\mu = (c\rho, \vec{J})$. $\qquad \left(S = \int d^4x\,\frac{1}{c}\,\mathcal{L}\right)$

$$\Rightarrow \boxed{\mathcal{L}_{int} = -\frac{1}{c}\,J_\mu A^\mu}$$

$\Rightarrow$ the full Lagrangian is

$$\boxed{\mathcal{L} = -\frac{1}{16\pi}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}\,J_\mu A^\mu}$$

Its Euler-Lagrange equations should give Maxwell equations: start by rewriting

$$\mathcal{L} = -\frac{1}{16\pi}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{c}\,J_\mu A^\mu$$