Last time

Gauge Invariance

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \Lambda(x)$$

$\Rightarrow F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} \Rightarrow \vec{E}$ & $\vec{B}$ fields are invariant under gauge transformations

$S_{int} = -\frac{1}{c^2}\int d^4x\, J_\mu A^\mu$ is also gauge-invariant.

Lagrangian for the Electromagnetic Field

Classical Mechanics		Classical Field Theory
q_i	$\to$	$\varphi(\vec{x}, t)$
i	$\to$	$\vec{x}, t$
$\dot{q}_i$	$\to$	$\partial_\mu \varphi(x)$
$L(q_i, \dot{q}_i)$	$\to$	$\int d^3x\, \mathcal{L}(\varphi, \partial_\mu \varphi)$

$\mathcal{L}$: Lagrangian density.

The action: $$\boxed{S = \frac{1}{c}\int d^4x\, \mathcal{L}(\varphi, \partial_\mu \varphi)}$$

EOM: $$\boxed{\frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu\left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)}\right] = 0}$$

$$\Rightarrow \quad \boxed{\frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta (\partial_\mu \varphi)} \right] = 0}$$

Euler-Lagrange equations for a field φ

cf. $\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$ in classical mechanics

To build the action for the E & M field require the following conditions on $\mathcal{L}_{EM}$:

(1) $\mathcal{L}_{EM}$ should be Lorentz-invariant

(2) $\mathcal{L}_{EM}$ should be gauge-invariant

(3) $\mathcal{L}_{EM}$ should be at most $\mathcal{O}(A_\mu{}^2)$.

$\Rightarrow$ the last condition is due to the superposition principle (experimental input) $\Rightarrow$ it leads to EOM being linear $\Rightarrow \mathcal{L}_{EM}$ should be at most quadratic in A_μ.

Lorentz invariants satisfying the above conditions are:

$$F_{\mu\nu} F^{\mu\nu}, \quad \underbrace{\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}}_{= -F_{\mu\nu} F^{\mu\nu} \text{ ← not new}}, \quad F_{\mu\nu} \tilde{F}^{\mu\nu}$$

However, $F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu$, with K^μ some 4-vector

$$\Rightarrow \int d^4x\, F_{\mu\nu}\tilde{F}^{\mu\nu} = \int d^4x\, \partial_\mu K^\mu = \underbrace{\int d\sigma_\mu\, K^\mu}_{\text{Surface of Space-time}} = 0 .$$

$\Rightarrow$ We conclude that $\mathcal{L} \propto F_{\mu\nu}F^{\mu\nu}$.

Note that rule (3) prevents terms like $F_{\mu\nu}F^{\nu\rho}F^\mu{}_\rho$ in the Lagrangian density.

$\Rightarrow$ Fixing normalization to match Gaussian units convention we write

$$\boxed{\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}}$$

Adding $\mathcal{L}_{int} = -\frac{1}{c} J_\mu A^\mu$ we get the net Lagrangian

$$\boxed{\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu}$$

(Note that if we add free particle Lagrangian then we'd get

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu - mc^2\sqrt{1 - \frac{v^2(t)}{c^2}}\ \delta^3(\vec{x} - \underbrace{\vec{x}(t)}_{\text{trajectory}})$$

$\Rightarrow$ vary $\vec{x}$, $\dot{\vec{x}}$ $\Rightarrow$ get Lorentz force

$$\frac{d\vec{p}}{dt} = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right].\ )$$

Euler - Lagrange equations (equations of motion) for A_μ are

$$\boxed{\frac{\delta \mathcal{L}}{\delta A_\mu} - \partial_\nu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} \right] = 0.}$$

$\Rightarrow$ use $\dfrac{\delta A_\mu}{\delta A_\nu} = \delta_\mu^{\ \nu}$, $\dfrac{\delta(\partial_\alpha A_\beta)}{\delta(\partial_\nu A_\mu)} = \delta_\alpha^{\ \nu} \delta_\beta^{\ \mu}$

$$\Rightarrow \frac{\delta \mathcal{L}}{\delta A_\mu} = -\frac{1}{c} J^\mu$$

as $\mathcal{L} = -\frac{1}{16\pi}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{c} J_\mu A^\mu$.

$$\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} = -\frac{1}{16\pi} \frac{\delta(F_{\alpha\beta} F^{\alpha\beta})}{\delta(\partial_\nu A_\mu)} = -\frac{1}{16\pi} g^{\alpha\rho} g^{\beta\sigma}.$$

$$\cdot \frac{\delta(F_{\alpha\beta} F_{\rho\sigma})}{\delta(\partial_\nu A_\mu)} = -\frac{1}{16\pi} g^{\alpha\rho} g^{\beta\sigma} \Big[(\delta_\alpha^{\ \nu} \delta_\beta^{\ \mu} - \delta_\alpha^{\ \mu} \delta_\beta^{\ \nu}) F_{\rho\sigma}$$

$$+ F_{\alpha\beta} (\delta_\rho^{\ \nu} \delta_\sigma^{\ \mu} - \delta_\rho^{\ \mu} \delta_\sigma^{\ \nu}) \Big] = -\frac{1}{16\pi} \left[F^{\nu\mu} - F^{\mu\nu} + F^{\nu\mu} - F^{\mu\nu} \right]$$

$$= \frac{1}{4\pi} F^{\mu\nu}$$

$\Rightarrow$ EOM are $-\frac{1}{c} J^\mu - \frac{1}{4\pi} \partial_\nu F^{\mu\nu} = 0$

$$\Rightarrow \boxed{\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu}$$

~ Maxwell equations (in Gaussian units) (2 out of 4)

$\mu = 0, 1, 2, 3 \Rightarrow 4$ equations

Remember the dual field-strength tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}.$$

Note that

$$\partial_\nu \tilde{F}^{\nu\mu} = \underbrace{\varepsilon^{\nu\mu\rho\sigma}}_{\text{anti-symm.}} \underbrace{\partial_\nu \partial_\rho}_{\text{symm.}} A_\sigma = 0$$

$$\Rightarrow \boxed{\partial_\nu \tilde{F}^{\nu\mu} = 0}$$

~ trivially follows from $\tilde{F}^{\mu\nu}$ definition

~ remaining 2 maxwell eqn's

$\mu = 0,1,2,3 \Rightarrow 4$ equations

To summarize, the traditional <u>Maxwell equations</u> are

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$

8 equations total

$\Rightarrow$ only the first equation has dynamical content

Maxwell Equations

Let us rewrite the obtained equations for $F_{\mu\nu}$ & $\tilde{F}_{\mu\nu}$ as equations for $\vec{E}$ & $\vec{B}$ fields.

Start with $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$.

$F^{0i} = -E^i$

$F^{ij} = -\varepsilon^{ijk} B^k$

$\boxed{\nu = 0}$ $\quad \partial_\mu F^{\mu 0} = \frac{4\pi}{c} J^0$

$\Rightarrow$ as $J^0 = c\rho$ and $\partial_\mu F^{\mu 0} = \partial_i F^{i0} = \partial_i E^i = \vec{\nabla} \cdot \vec{E}$

(since $\partial_i = \nabla^i$) $\Rightarrow$ get

$$\boxed{\vec{\nabla} \cdot \vec{E} = 4\pi\rho}$$

Coulomb's Law

ca. 1785 (aka Gauss's Law)

$\boxed{\nu = i}$ $\quad \partial_\mu F^{\mu i} = \frac{4\pi}{c} J^i \quad , i = 1, 2, 3$

$$\Rightarrow \partial_0 F^{0i} + \partial_j F^{ji} = \frac{4\pi}{c} J^i$$

$$-\partial_0 E^i - \partial_j \varepsilon^{jik} B^k = \frac{4\pi}{c} J^i$$

$$\underbrace{\varepsilon^{ijk} \nabla^j B^k}_{(\vec{\nabla} \times B)^i} = \partial_0 E^i + \frac{4\pi}{c} J^i$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}}$$

1826 1861

Ampere–Maxwell Law

$\tilde{F}^{\mu\nu}$ has the following components:

$$\tilde{F}^{0i} = \frac{1}{2}\varepsilon^{0ijk}F_{jk} = \frac{1}{2}\varepsilon^{ijk}(-)\varepsilon^{jkl}B^l$$

$\Rightarrow$ use $\varepsilon^{ijk}\varepsilon^{ij'k'} = \delta^{jj'}\delta^{kk'} - \delta^{jk'}\delta^{kj'}$ to write

$$\varepsilon^{ijk}\varepsilon^{ijk'} = 2\delta^{kk'} \Rightarrow \tilde{F}^{0i} = -\frac{1}{2}\cdot 2\delta^{il}B^l = -B^i$$

$$\Rightarrow \boxed{\tilde{F}^{0i} = -B^i}$$

$$\tilde{F}^{ij} = \frac{1}{2}\varepsilon^{ij\rho\sigma}F_{\rho\sigma} = \varepsilon^{ij0k}F_{0k} = -\varepsilon^{ijk}F^{0k} =$$

$$= -\varepsilon^{ijk}(-E^k) = \varepsilon^{ijk}E^k \Rightarrow \boxed{\tilde{F}^{ij} = \varepsilon^{ijk}E^k}$$

$\Rightarrow$ let's study $\partial_\mu \tilde{F}^{\mu\nu} = 0$

$\boxed{\nu = 0}$ $\quad \partial_\mu \tilde{F}^{\mu 0} = 0 \Rightarrow \partial_i \tilde{F}^{i0} = 0 \Rightarrow \partial_i B^i = 0$

$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$ ~ law stating the absence of magnetic monopoles

$\boxed{\nu = i}$ $\quad \partial_\mu \tilde{F}^{\mu i} = 0 \Rightarrow \partial_0 \tilde{F}^{0i} + \partial_j \tilde{F}^{ji} = 0$

$$\Rightarrow -\partial_0 B^i + \partial_j \varepsilon^{jik}E^k = 0 \Rightarrow \underbrace{\varepsilon^{ijk}\nabla^j E^k}_{(\vec{\nabla}\times\vec{E})^i} + \underbrace{\partial_0 B^i}_{\frac{1}{c}\frac{\partial B^i}{\partial t}} = 0$$

$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0}$ <u>Faraday's Law</u>, ca. 1831