Last time | We constructed the Lagrangian density for the electromagnetic field. Combined with the interaction term for charges, we had:

$$\boxed{\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu}$$

We derived the EOM for this Lagrangian density:

Maxwell equations:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$ ← EOM proper

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$ ← always true for $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

Maxwell Equations

$\boxed{\nu = 0}$ in the 1st eq'n: $\boxed{\vec{\nabla} \cdot \vec{E} = 4\pi\rho}$ Coulomb's Law (Gauss's Law)

$\boxed{\nu = i}$ — " — : $\boxed{\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}}$ Ampere-Maxwell Law

Maxwell Equations

Let us rewrite the obtained equations for $F_{\mu\nu}$ & $\tilde{F}_{\mu\nu}$ as equations for $\vec{E}$ & $\vec{B}$ fields.

Start with $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$.

$F^{0i} = -E^i$

$F^{ij} = -\varepsilon^{ijk} B^k$

$\boxed{\nu = 0}$ $\quad \partial_\mu F^{\mu 0} = \frac{4\pi}{c} J^0$

$\Rightarrow$ as $J^0 = c\rho$ and $\partial_\mu F^{\mu 0} = \partial_i F^{i0} = \partial_i E^i = \vec{\nabla} \cdot \vec{E}$

(since $\partial_i = \nabla^i$) $\Rightarrow$ get

$$\boxed{\vec{\nabla} \cdot \vec{E} = 4\pi\rho}$$

Coulomb's Law

ca. 1785 (aka Gauss's Law)

$\boxed{\nu = i}$ $\quad \partial_\mu F^{\mu i} = \frac{4\pi}{c} J^i \quad , i = 1, 2, 3$

$$\Rightarrow \partial_0 F^{0i} + \partial_j F^{ji} = \frac{4\pi}{c} J^i$$

$$-\partial_0 E^i - \partial_j \varepsilon^{jik} B^k = \frac{4\pi}{c} J^i$$

$$\underbrace{\varepsilon^{ijk} \nabla^j B^k}_{(\vec{\nabla} \times B)^i} = \partial_0 E^i + \frac{4\pi}{c} J^i$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}}$$

1826 1861

Ampere-Maxwell Law

$\tilde{F}^{\mu\nu}$ has the following components:

$$\tilde{F}^{0i} = \frac{1}{2}\varepsilon^{0ijk}F_{jk} = \frac{1}{2}\varepsilon^{ijk}(-)\varepsilon^{jkl}B^l$$

$\Rightarrow$ use $\varepsilon^{ijk}\varepsilon^{ij'k'} = \delta^{jj'}\delta^{kk'} - \delta^{jk'}\delta^{kj'}$ to write

$$\varepsilon^{ijk}\varepsilon^{ijk'} = 2\delta^{kk'} \Rightarrow \tilde{F}^{0i} = -\frac{1}{2}\cdot 2\delta^{il}B^l = -B^i$$

$$\Rightarrow \boxed{\tilde{F}^{0i} = -B^i}$$

$$\tilde{F}^{ij} = \frac{1}{2}\varepsilon^{ij\rho\sigma}F_{\rho\sigma} = \varepsilon^{ij0k}F_{0k} = -\varepsilon^{ijk}F^{0k} =$$

$$= -\varepsilon^{ijk}(-E^k) = \varepsilon^{ijk}E^k \Rightarrow \boxed{\tilde{F}^{ij} = \varepsilon^{ijk}E^k}$$

$\Rightarrow$ let's study $\partial_\mu \tilde{F}^{\mu\nu} = 0$

$\boxed{\nu = 0}$ $\quad \partial_\mu \tilde{F}^{\mu 0} = 0 \Rightarrow \partial_i \tilde{F}^{i0} = 0 \Rightarrow \partial_i B^i = 0$

$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$ ~ law stating the absence of magnetic monopoles

$\boxed{\nu = i}$ $\quad \partial_\mu \tilde{F}^{\mu i} = 0 \Rightarrow \partial_0 \tilde{F}^{0i} + \partial_j \tilde{F}^{ji} = 0$

$$\Rightarrow -\partial_0 B^i + \partial_j \varepsilon^{jik}E^k = 0 \Rightarrow \underbrace{\varepsilon^{ijk}\nabla^j E^k}_{(\vec{\nabla}\times\vec{E})^i} + \underbrace{\partial_0 B^i}_{\frac{1}{c}\frac{\partial B^i}{\partial t}} = 0$$

$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0}$ <u>Faraday's Law</u>, ca. 1831

To summarize, in 3-vector notation we have the following Maxwell equations (ca. 1865):

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = 4\pi\rho & \qquad & \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \\ &\vec{\nabla}\cdot\vec{B} = 0 & & \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \end{aligned}}$$

Gaussian Units

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ ~ Coulomb's Law (Gauss's Law)

$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$ ~ Ampere-Maxwell Law

$\vec{\nabla}\cdot\vec{B} = 0$ ~ no monopoles

$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$ ~ Faraday's Law

For future use, let us rewrite Maxwell equations in SI (Le Système International d'unités):

(see page 781 of Jackson for a useful table)

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 & \qquad & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \\ &\vec{\nabla}\cdot\vec{B} = 0 & & \vec{\nabla}\times\vec{E} + \cancel{\frac{1}{c}}\frac{\partial\vec{B}}{\partial t} = 0 \end{aligned}}$$

Maxwell equations, SI units

$\frac{1}{\mu_0\varepsilon_0} = c^2$, $\varepsilon_0 \simeq 8.854 \times 10^{-12}$ F/m, 1 Farad = 1 $\frac{\text{Coulomb}^2}{\text{N}\cdot\text{m}}$

$\mu_0 = 4\pi \times 10^{-7} \frac{\text{Newtons}}{\text{Ampere}^2}$, 1 Ampere = $\frac{1\ \text{Coulomb}}{\text{Second}}$

We have obtained Maxwell equations by assuming

- Lorentz-invariance of physics
- existence of a vector field $A^\mu(x)$
- gauge-invariance of physics (charge conservation)

⊕ superposition principle for $A^\mu(x)$.

Conservation Laws and Energy-Momentum Tensor

Noether's theorem: Every continuous symmetry of the action corresponds to a conservation law.

Space-time is homogeneous $\Rightarrow$ physics is invariant under space-time translations

$$x^\mu \to x^\mu - a^\mu \quad , \quad a^\mu = \text{const}$$

$\Rightarrow$ energy & momentum are conserved

$\Rightarrow$ let's find energy and momentum of an E & M - interacting system

Conservation Laws:

Energy conservation in mechanics:

$S = \int dt\, L(q_i, \dot{q}_i, t) \Rightarrow$ EL equations read

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 .$$

Now,
$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} = (\text{EL eqns})$$

$$= \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right] + \frac{\partial L}{\partial t} =$$

$$= \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L}_{\substack{\shortparallel \\ E}}\Big) = -\frac{\partial L}{\partial t}$$

E energy of the system (cf. Hamiltonian)

If $L = L(q_i, \dot{q}_i) \Rightarrow \frac{\partial L}{\partial t} = 0 \Rightarrow \boxed{\frac{dE}{dt} = 0}$

$\Rightarrow$ energy is conserved!

Imagine a field theory with

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi) \qquad (\text{no } x^\mu\text{-dependence in } \mathcal{L})$$

Imagine an infinitesimal space-time shift:

$$x^\mu \to x^\mu - \delta a^\mu = x'^\mu \Rightarrow x^\mu = x'^\mu + \delta a^\mu$$

ϕ is inv. $\phi'(x') = \phi(x)$

$$\Rightarrow \phi(x) \to \phi(x'^\mu + \delta a^\mu) \simeq \phi(x'^\mu) + \overbrace{\delta a^\mu \partial_\mu \phi(x')}^{\delta\phi}$$

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \phi} \delta\phi + \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) = \underbrace{\left[\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \right) \right]}_{\overset{\text{(EOM)}}{=} 0} \delta\phi$$

$$+ \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi \right)$$

$$\Rightarrow \delta \mathcal{L} = \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta a^\nu \partial_\nu \phi \right).$$

On the other hand $\mathcal{L}$ is scalar $\Rightarrow \mathcal{L} = \mathcal{L}(x) \Rightarrow$

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta a^\mu \partial_\mu \mathcal{L} \Rightarrow \delta \mathcal{L} = \delta a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

Equating two $\delta\mathcal{L}$'s we get

$$\delta a^\nu \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right] = 0 \quad \text{for any } \delta a^\nu$$

Def: Energy-momentum tensor

$$\boxed{T^{\mu}{}_{\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^{\mu}{}_{\nu} \mathcal{L}}$$

$\Rightarrow$ $\partial_\mu T^{\mu}{}_{\nu} = 0$ conserved!