Last time | We finished deriving Maxwell Equations for $\vec{E}$ & $\vec{B}$ fields:

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 4\pi\rho \qquad & \vec{\nabla}\times\vec{B} &= \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} &= 0 \qquad & \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} &= 0 \end{aligned}}$$

Conservation Laws and the Energy-Momentum Tensor (cont'd)

Noether's theorem: every continuous symmetry of the action $\Leftrightarrow$ conservation law

$x^\mu \to x'^\mu = x^\mu - \delta a^\mu$ ~ physics is invariant under space-time translations

$\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$ ~ some Lagrangian density for field $\varphi(x)$

$$\varphi(x) \to \varphi'(x') = \varphi(x) = \varphi(x' + \delta a) = \varphi(x') + \delta a^\mu \partial_\mu \varphi(x')$$

$$\Rightarrow \Delta\varphi = \varphi'(x') - \varphi(x') = \delta a^\mu \partial_\mu \varphi(x')$$

Varying the Lagrangian density and using EOM we obtained the following conservation law:

Def. Energy-momentum tensor:

$$\boxed{T^\mu{}_\nu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^\mu{}_\nu \mathcal{L}}$$

where $\boxed{\partial_\mu T^\mu{}_\nu = 0}$ ~ a conserved quantity

(cf. $\partial_\mu J^\mu = 0$)

We have derived a tensor which is explicitly conserved: $\boxed{\partial_\mu T^{\mu\nu} = 0}$

Apply these results to EM: $\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\mu\nu}^2$

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L}_{EM}$$

$$\Rightarrow T^{\mu\nu}_{EM} = \frac{1}{4\pi} F^{\rho\mu} \partial^\nu A_\rho + \frac{1}{16\pi} g^{\mu\nu} F_{\rho\sigma}^2$$

However, this definition of energy-momentum tensor is not unique, in the sense that one can always add $T^{\mu\nu} \to T^{\mu\nu} + \partial_\rho \psi^{\rho\mu\nu}$,

where $\psi^{\rho\mu\nu}$ is some anti-symmetric tensor with $\psi^{\rho\mu\nu} = -\psi^{\mu\rho\nu} \Rightarrow$

$\Rightarrow$ then $\partial_\mu T^{\mu\nu} \to \partial_\mu T^{\mu\nu} + \left(\partial_\mu \partial_\rho \psi^{\rho\mu\nu} = 0\right)$

$\Rightarrow$ can use this property to define a symmetric energy-momentum tensor:

$$T^{\mu\nu} = T^{\nu\mu}$$

To fix the definition of $T_{\mu\nu}$ let's require conservation of total angular momentum of the field:

(cf. problem 6.10 in Jackson):

$$\vec{\mathcal{L}}_{field} = \frac{1}{4\pi c} \vec{x} \times (\vec{E} \times \vec{B})$$

← Gauss units

$\Rightarrow$ one can show that $\frac{\partial}{\partial t} \int_V \mathcal{L}^i_{field} \, d^3x + \int_S da \cdot n_j M_{ji} = 0$

where $M_{ijk} = T_{ij} x_k - T_{ik} x_j$ and $M_{ij} = \varepsilon_{jkl} \frac{1}{2} M_{kil}$

(T_{ij} was Maxwell's stress tensor ~ just ij components of $T_{\mu\nu}$)

$\Rightarrow$ as could be shown the above conservation law is $\partial_\mu M^{\mu\nu\rho} = 0$ where

$$M^{\mu\nu\rho} \equiv T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu$$

$\Rightarrow$ if we want $\partial_\mu M^{\mu\nu\rho} = 0 \Rightarrow 0 = \cancel{\partial_\mu T^{\mu\nu}}^{\to 0} \cdot x^\rho -$

$- \cancel{\partial_\mu T^{\mu\rho}} x^\nu{}^{\to 0} + \quad T^{\rho\nu} - T^{\nu\rho}$

$\Rightarrow \partial_\mu M^{\mu\nu\rho} = 0$ requires $T^{\rho\nu} = T^{\nu\rho}$. $\Rightarrow$

To symmetrize $T^{\mu\nu}_{EM}$ subtract $\frac{1}{4\pi}\partial_\rho(F^{\rho\mu}A^\nu)$:

$$T^{\mu\nu}_{symm} = \frac{1}{4\pi} F^{\rho\mu}\partial^\nu A_\rho - \frac{1}{4\pi}\partial_\rho(F^{\rho\mu}A^\nu) - g^{\mu\nu}\mathcal{L}_{EM}$$

$$= \frac{1}{4\pi} F^{\rho\mu}\partial^\nu A_\rho - \frac{1}{4\pi}\underbrace{\cancel{\partial_\rho F^{\rho\mu}}}_{\to 0 \text{ (Maxwell)}} A^\nu - \frac{1}{4\pi} F^{\rho\mu}\partial_\rho A^\nu -$$

$$-g^{\mu\nu}\mathcal{L}_{EM} = \frac{1}{4\pi} F^{\rho\mu} F^\nu{}_\rho - g^{\mu\nu}\mathcal{L}_{EM}$$

$$\Rightarrow T^{\mu\nu} = -\frac{1}{4\pi} F^{\mu\rho} F^\nu{}_\rho + \frac{1}{16\pi} g^{\mu\nu} F_{\mu\nu}{}^2$$

$$\Rightarrow \boxed{T^{\mu\nu} = \frac{1}{4\pi}\left(-F^{\mu\rho}F^\nu{}_\rho + \frac{g^{\mu\nu}}{4} F_{\alpha\beta}F^{\alpha\beta}\right)}$$

Properties of $T^{\mu\nu}$:

$$T^\mu{}_\mu = \frac{1}{4\pi}\left(-F^{\mu\rho}F_{\mu\rho} + \frac{4}{4}F_{\alpha\beta}F^{\alpha\beta}\right) = 0$$

$\Rightarrow$ <u>traceless</u>!

$$T^{00} = \frac{1}{4\pi}\left(-F^{0i}F^0{}_i + \frac{1}{4}\underbrace{F_{\mu\nu}^2}_{2(B^2-E^2)}\right) = \frac{1}{8\pi}(B^2-E^2) +$$

$$+\frac{1}{4\pi}E^2 = \frac{1}{8\pi}(B^2+E^2) \sim \text{energy density}$$

(in Gaussian units)

$$T^{0i} = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)_i$$ ~ momentum density

$$T^{ij} = \frac{-1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}\left(E^2 + B^2\right)\right]$$

$= (-)$ Maxwell's stress tensor

$$\Rightarrow T^{\mu\nu} = \left(\begin{array}{c|c} \text{Energy density} & \text{momentum density} \\ \hline \text{momentum density} & -\text{Maxwell's stress tensor} \end{array}\right)$$

$\partial_\mu T^{\mu\nu} = 0$ ~ energy & momentum conservation.