Last time | Used the general formula

$$\boxed{T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L}}$$

to construct E&M field's energy-momentum tensor: (EMT)

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\rho\mu} \partial^\nu A_\rho + \frac{1}{16\pi} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

a trick:

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\rho \psi^{\rho\mu\nu}$$

where $\psi^{\rho\mu\nu} = -\psi^{\mu\rho\nu}$ allows one to symmetrize the EMT:

$$\boxed{T^{\mu\nu}_{EM} = \frac{1}{4\pi}\left[-F^{\mu\rho} F^{\nu}{}_{\rho} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right]}$$

while having it conserved still:

$$\boxed{\partial_\mu T^{\mu\nu}_{EM} = 0}$$

Consider E&M Lagrangian including the charges:

$$\boxed{\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu}$$

It has an explicit dependence on current $J_\mu(x)$.

Hence under $x^\mu \to x'^\mu = x^\mu - \delta a^\mu$ we get

$$A_\mu(x) \to A'_\mu(x') = A_\mu(x) = A_\mu(x'^\nu + \delta a^\nu) = A_\mu(x') + \delta a^\nu \partial_\nu A_\mu(x') + \ldots$$

$$J_\mu(x) \to J'_\mu(x') = J_\mu(x) = J_\mu(x'^\nu + \delta a^\nu) = J_\mu(x') + \delta a^\nu \partial_\nu J_\mu(x') + \ldots$$

$$\delta\mathcal{L} = \underbrace{\frac{\delta\mathcal{L}}{\delta A_\mu}\delta A_\mu + \frac{\delta\mathcal{L}}{\delta(\partial_\nu A_\mu)}\delta(\partial_\nu A_\mu)} + \frac{\delta\mathcal{L}}{\delta J_\mu}\delta J_\mu = \delta a^\rho \partial_\nu(\delta^\nu_\rho \mathcal{L})$$

use EOM $\to$ $\partial_\nu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\nu A_\mu)}\delta a^\rho \partial_\rho A_\mu\right)$

$$\Rightarrow \quad \delta a^\rho \partial_\nu \Big[\underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_\nu A_\mu)}\partial_\rho A_\mu - \delta^\nu_\rho \mathcal{L}}_{T^\nu{}_\rho}\Big] = -\frac{\delta\mathcal{L}}{\delta J_\mu}\delta a^\rho \partial_\rho J_\mu$$

$$\Rightarrow \boxed{\partial_\nu T^\nu{}_\rho = \underbrace{-\frac{\delta\mathcal{L}}{\delta J_\mu}\partial_\rho J_\mu}_{\frac{1}{c}A^\mu \partial_\rho J_\mu}}$$ ~ energy-momentum tensor is not conserved.

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\rho\mu}\partial^\nu A_\rho + \frac{1}{16\pi} g^{\mu\nu} F_{\rho\sigma}^2 + g^{\mu\nu}\frac{1}{c}J_\rho A^\rho$$

Again, subtract $\frac{1}{4\pi}\partial_\rho(F^{\rho\mu}A^\nu)$ $\Rightarrow$

$$T^{\mu\nu}_{\text{symm}} = \frac{1}{4\pi} F^{\rho\mu} \partial^\nu A_\rho - \frac{1}{4\pi} \underbrace{\partial_\rho F^{\rho\mu}}_{\frac{4\pi}{c} J^\mu} A^\nu - \frac{1}{4\pi} F^{\rho\nu} \partial_\rho A^\nu$$

~ now have sources

$$- g^{\mu\nu} \mathcal{L}_{EM} = -\frac{1}{4\pi} F^{\mu\rho} F^\nu{}_\rho + g^{\mu\nu} \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} - \frac{1}{c} J^\mu A^\nu + g^{\mu\nu} \frac{1}{c} J_\rho A^\rho$$

$$\Rightarrow \partial_\mu \left[-\frac{1}{4\pi} F^{\mu\rho} F^\nu{}_\rho + g^{\mu\nu} \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} + \frac{1}{c} g^{\mu\nu} J_\rho A^\rho \right] - \frac{1}{c} \partial_\mu \left(J^\mu A^\nu \right) =$$

$$= \frac{1}{c} A^\mu \partial^\nu J_\mu$$

$\Rightarrow$ again defining $T^{\mu\nu}_{EM} = -\frac{1}{4\pi} F^{\mu\rho} F^\nu{}_\rho + g^{\mu\nu} \frac{1}{16\pi} F_{\rho\sigma} F^{\rho\sigma}$

and using $\partial_\mu J^\mu = 0$ we get

$$\partial_\mu T^{\mu\nu}_{EM} = \frac{1}{c} J^\mu \partial_\mu A^\nu - \frac{1}{c} J^\mu \partial^\nu A_\mu$$

$$\Rightarrow \boxed{\partial_\mu T^{\mu\nu}_{EM} = \frac{1}{c} J_\mu F^{\mu\nu}}$$

put $\boxed{\nu = 0}$: $\partial_0 T^{00}_{EM} + \partial_i T^{i0}_{EM} = \frac{1}{c} J_i \underbrace{F^{i0}}_{E^i} = -\frac{1}{c} \vec{J} \cdot \vec{E}$

$$\Rightarrow \frac{1}{c} \frac{\partial}{\partial t} T^{00}_{EM} + \nabla^i T^{i0}_{EM} = -\frac{1}{c} \vec{J} \cdot \vec{E}$$

~ this looks like a conservation law

(cf. $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$), but the r.h.s. $\neq 0$.

(Def.) $u \equiv \dfrac{E^2+B^2}{8\pi} = T^{00}_{EM}$ (energy density)

(Def.) $\vec{S} \equiv \dfrac{c}{4\pi} \vec{E} \times \vec{B}$ ~ Poynting vector

$$\frac{S^i}{c} = T^{0i}_{EM} \quad \text{as} \quad T^{0i}_{EM} = \frac{1}{4\pi}\left(-F^{0\rho} F^i{}_\rho\right) =$$

$$= + \frac{1}{4\pi} \underbrace{F^{0j}}_{-E^j} \underbrace{F^{ij}}_{-\varepsilon^{ijk} B^k} = \frac{1}{4\pi} \varepsilon^{ijk} E^j B^k = \frac{1}{4\pi} \left(\vec{E} \times \vec{B}\right)^i$$

=> plugging into the above equation get

$$\frac{1}{c} \frac{\partial}{\partial t} u + \frac{\vec{\nabla} \cdot \vec{S}}{c} = -\frac{1}{c} \vec{J} \cdot \vec{E}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}}$$ Poynting's theorem

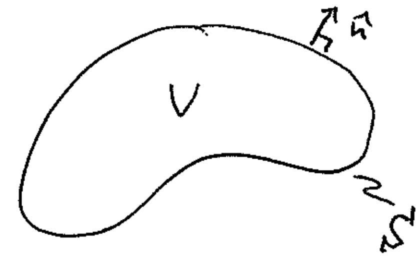
$\vec{J} \cdot \vec{E}$ = rate of E&M energy change due to work done on charges ($\vec{J} = q\vec{v} \Rightarrow \vec{J} \cdot \vec{E} = q \vec{v} \cdot \vec{E} = \frac{d\mathcal{E}}{dt}$ for a single point charge)

=> write $\vec{J} \cdot \vec{E} = \dfrac{\partial u_{mech}}{\partial t}$ => Poynting's th'm

becomes $$\boxed{\frac{\partial u_{field}}{\partial t} + \frac{\partial u_{mech}}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0}$$

Consider a volume V and integrate over it: (74)

$$\int_V d^3x \left[\frac{\partial u_{field}}{\partial t} + \frac{\partial u_{mech}}{\partial t} \right] = -\int_V d^3x\ \vec{\nabla}\cdot\vec{S}$$



Using divergence theorem get

$$\frac{d\mathcal{E}_{field}}{dt} + \frac{d\mathcal{E}_{mech}}{dt} = -\oint_S da\ \hat{n}\cdot\vec{S}$$ ~ flow of energy in/out of the system

where $\mathcal{E} = \int_V d^3x\, u$ ~ energy in V.

(Assuming no particles move in/out of the volume.)

$\Rightarrow$ Poynting vector has the meaning of energy flow

Get back to $\partial_\mu T^{\mu\nu}_{EM} = \frac{1}{c} J_\mu F^{\mu\nu}$.

Put $\boxed{\nu = i}$: $\quad \partial_0 T^{0i}_{EM} + \partial_j T^{ji}_{EM} = \frac{1}{c} J_0 F^{0i} + \frac{1}{c} J_j F^{ji}$

(Def) Momentum of the field by

$$\boxed{\vec{P}_{field} \equiv \int d^3x\ \frac{1}{c^2}\vec{S} = \int d^3x\ \frac{1}{4\pi c}\vec{E}\times\vec{B}} = \int d^3x\ \frac{1}{c}\, T^{0i}_{EM}$$

Integration over volume V gives:

$$\frac{1}{c}\frac{d}{dt}\left(\vec{P}_{field}\right)^i = \int_V d^3x \left[-\partial_j T^{ji}_{EM} + \rho F^{0i} + \frac{1}{c} J_j F^{ji} \right]$$