Last time | Conservation Laws and the EMT (cont'd)

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$$

$\Rightarrow$ obtained the following law:

$$\boxed{\partial_\mu T^{\mu\nu}_{EM} = \frac{1}{c} J_\mu F^{\mu\nu}}$$

where

$$T^{\mu\nu}_{EM} = -\frac{1}{4\pi} F^{\mu\rho}F^{\nu}{}_{\rho} + g^{\mu\nu}\frac{1}{16\pi} F_{\rho\sigma}F^{\rho\sigma}$$

Started with the $\boxed{\nu = 0}$ component of the conservation law: obtained Poynting's th'm,

$$\boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}}$$

where

$$\boxed{u \equiv \frac{E^2 + B^2}{8\pi} = T^{00}_{EM}}$$ is the energy density,

and

$$\boxed{\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}}$$ is the Poynting vector.

(energy flow)

$\vec{J}\cdot\vec{E}$ = rate of E&M energy change due to work done on charges = $\frac{\partial U_{mech}}{\partial t}$

$\Rightarrow$ got energy conservation as

$$\boxed{\frac{\partial u_{field}}{\partial t} + \frac{\partial u_{mech}}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0}$$

Def. $u \equiv \frac{E^2+B^2}{8\pi} = T^{00}_{EM}$ (energy density)

Def. $\vec{S} \equiv \frac{c}{4\pi} \vec{E} \times \vec{B} \sim$ Poynting vector

$$\frac{S^i}{c} = T^{0i}_{EM} \quad \text{as} \quad T^{0i}_{EM} = \frac{1}{4\pi}\left(-F^{0\rho} F^{i}{}_{\rho}\right) =$$

$$= + \frac{1}{4\pi} \underbrace{F^{0j}}_{-E^j} \underbrace{F^{ij}}_{-\varepsilon^{ijk} B^k} = \frac{1}{4\pi} \varepsilon^{ijk} E^j B^k = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)^i$$

$\Rightarrow$ plugging into the above equation get

$$\frac{1}{c}\frac{\partial}{\partial t} u + \frac{\vec{\nabla}\cdot\vec{S}}{c} = -\frac{1}{c}\vec{J}\cdot\vec{E}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}}$$ Poynting's theorem

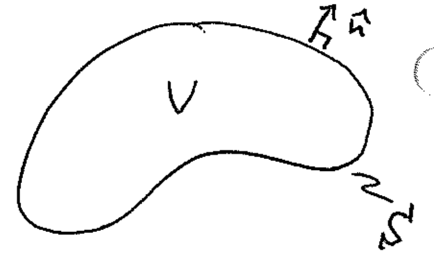
$\vec{J}\cdot\vec{E}$ = rate of E&M energy change due to work done on charges ($\vec{J} = q\vec{v} \Rightarrow \vec{J}\cdot\vec{E} = q\vec{v}\cdot\vec{E} = \frac{d\varepsilon}{dt}$ for a single point charge)

$\Rightarrow$ write $\vec{J}\cdot\vec{E} = \frac{\partial u_{mech}}{\partial t}$ $\Rightarrow$ Poynting's th'm becomes

$$\boxed{\frac{\partial u_{field}}{\partial t} + \frac{\partial u_{mech}}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0}$$

Consider a volume V and integrate over it: (74)

$$\int_V d^3x \left[\frac{\partial u_{field}}{\partial t} + \frac{\partial u_{mech}}{\partial t} \right] = -\int_V d^3x \, \vec{\nabla} \cdot \vec{S}$$



Using divergence theorem get

$$\frac{d\mathcal{E}_{field}}{dt} + \frac{d\mathcal{E}_{mech}}{dt} = -\oint_S da \, \hat{n} \cdot \vec{S}$$ ~ flow of energy in/out of the system

where $\mathcal{E} = \int_V d^3x \, u$ ~ energy in V.

(Assuming no particles move in/out of the volume.)

$\Rightarrow$ Poynting vector has the meaning of energy flow (out of the system)

Get back to $\partial_\mu T^{\mu\nu}_{EM} = \frac{1}{c} J_\mu F^{\mu\nu}$.

Put $\boxed{\nu = i}$: $\partial_0 T^{0i}_{EM} + \partial_j T^{ji}_{EM} = \frac{1}{c} J_0 F^{0i} + \frac{1}{c} J_j F^{ji}$

(Def) Momentum of the field by

$$\boxed{\vec{P}_{field} \equiv \int d^3x \, \frac{1}{c^2} \vec{S} = \int d^3x \, \frac{1}{4\pi c} \vec{E} \times \vec{B}} = \int d^3x \, \frac{1}{c} T^{0i}_{EM}$$

Integration over volume V gives:

$$\frac{d}{dt}\left(\vec{P}_{field}\right)^i = \int_V d^3x \left[-\partial_j T^{ji}_{EM} + \rho F^{0i} + \frac{1}{c} J_j F^{ji} \right]$$

$$T^{ij}_{EM} = -\frac{1}{4\pi} F^{i\rho} F^{j}{}_{\rho} - \frac{\delta^{ij}}{16\pi} \underbrace{F_{\rho\sigma} F^{\rho\sigma}}_{2(B^2-E^2)} =$$

$$= -\frac{1}{4\pi}\Big(\underset{E^i}{\underset{\shortparallel}{F^{i0}}}\ \underset{E^j}{\underset{\shortparallel}{F^{j}{}_{0}}} - \underset{-\varepsilon^{ikl}B^l}{\underset{\shortparallel}{F^{ik}}}\ \underset{-\varepsilon^{jkl'}B^{l'}}{\underset{\shortparallel}{F^{jk}}}\Big) - \frac{1}{16\pi}\delta^{ij}\cdot 2(B^2-E^2) =$$

$$= -\frac{1}{4\pi}\Big[E^i E^j - \underbrace{\varepsilon^{ikl}\varepsilon^{jkl'}}_{\delta^{ij}\delta^{ll'} - \delta^{il'}\delta^{jl}} B^l B^{l'} + \frac{1}{2}\delta^{ij}(B^2-E^2)\Big]$$

$$= -\frac{1}{4\pi}\Big[E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(B^2+E^2)\Big]$$

$$\Rightarrow \boxed{T^{ij}_{EM} = -\frac{1}{4\pi}\Big[E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(B^2+E^2)\Big]}$$

Def. Maxwell stress tensor $\sigma^{ij} = -T^{ij}_{EM}$

$$\frac{d}{dt} P^i_{field} = \int_V d^3x \Big[\nabla^j \sigma^{ji} + \underbrace{\rho \underbrace{F^{0i}}_{-E^i} + \frac{1}{c} J_j \underbrace{F^{ji}}_{-\varepsilon^{jik}B^k}}_{-\left[\rho E^i + \frac{\varepsilon^{ijk}}{c} J^j B^k\right]}\Big]$$

$$\frac{d P^i_{field}}{dt} = \int_V d^3x\, \nabla^j \sigma^{ji} - \int d^3x \Big[\rho E^i + \Big(\frac{\vec{J}\times\vec{B}}{c}\Big)^i\Big]$$

Lorentz force: $\dfrac{d\vec{p}}{dt} = q\vec{E} + q\dfrac{\vec{v}}{c}\times\vec{B}$

$\Rightarrow$ for continuous distribution of charge have

$$\frac{d\vec{p}_{mech}}{dt} = \int d^3x \left[\rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}\right]$$

$\Rightarrow$ momentum conservation reads:

$$\boxed{\frac{d}{dt}\left(p^i_{field} + p^i_{mech}\right) = \int_V d^3x\, \nabla^j \sigma^{ij} = \oint_S da\, n^j \sigma^{ij}}$$

$\Rightarrow \sigma^{ij}$ has the meaning of

momentum flow into the system

$\hookrightarrow$ flow of p^i in direction $\hat{x}^j$.

Note that

$$\mathrm{Tr}\,\sigma^{ij} = \sigma^{ii} = \frac{1}{4\pi}\left[\vec{E}^2 + \vec{B}^2 - \frac{3}{2}(E^2+B^2)\right]$$

$$= -\frac{1}{8\pi}\left[E^2+B^2\right] = -u \quad \sim \text{energy density.}$$

$$\Rightarrow T_\mu{}^\mu = T_0{}^0 + T_i{}^i = T^{00} - T^{ii} = T^{00} + \sigma^{ii} =$$

$= u - u = 0 \quad \Rightarrow T^{\mu\nu}$ is traceless!

<u>Energy - Momentum tensor:</u>

we studied: u ~ energy density

$\vec{S}$ ~ Poynting vector (energy flow)

σ_{ij} ~ Maxwell stress tensor (momentum flow)

These seemingly unrelated quantities form one tensor under Lorentz transformations, the energy-momentum tensor:

$$T^{\mu\nu} = \begin{pmatrix} u & S^1/c & S^2/c & S^3/c \\ S^1/c & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ S^2/c & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ S^3/c & -\sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{pmatrix}$$

where $\mu, \nu = 0, 1, 2, 3$

0 ~ time component: $x^0 = tc$, $x^1 = x$, $x^2 = y$, $x^3 = z$

all the above conservation laws read

$$\boxed{\frac{\partial}{\partial x^\mu} T^{\mu\nu}_{EM} = \frac{1}{c} J_\mu F^{\mu\nu}}$$ (summation over μ)

$$T_\mu{}^\mu = u + T_i{}^i = u - u = 0 \sim \text{traceless}$$

T_{11}, T_{22}, T_{33} ~ radiation pressure components

(T_{ij} is the flux of the ith component of momentum density in the $\hat{j}$ direction)

Classification of Physical Quantities by Space-Time Symmetries.

A. Spatial rotations

$i, j = 1, 2, 3$

$$X_i \to X'_i = R_{ij} X_j \quad , \quad R_{ij} \text{ - rotation matrix}$$

$$\vec{X}'^2 = \vec{X}^2 \text{ under rotation} \Rightarrow (R_{ij} X_j)(R_{ik} X_k) = X_i X_i$$

$$\Rightarrow R_{ij} R_{ik} = \delta_{jk} \quad \Rightarrow \quad (R^{-1})_{ij} = R_{ji}$$

$$\Rightarrow (\det R)^2 = 1$$

$\det R = +1$ ~ rotation w/o reflection ($\det R = -1$: rot + reflection)

vectors: $A_i \to A'_i = R_{ij} A_j$

tensors: $T_{i_1 i_2 \dots i_n} \to T'_{i_1 i_2 \dots i_n} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_n j_n} \, T_{j_1 j_2 \dots j_n}$

(definition) n = rank of the tensor

B. Spatial Reflection (parity)

$\vec{X} \to -\vec{X}$ ~ all vectors (or polar vectors) transform like this

$$\vec{z} = \vec{x} \times \vec{y} \Rightarrow \left.\begin{array}{l} \vec{x} \to -\vec{x} \\ \vec{y} \to -\vec{y} \end{array}\right\} \Rightarrow \vec{z} \to \vec{z}$$

axial vector (pseudovector)