Last time | Def. $\vec{P}_{field} \equiv \int d^3x \, \frac{1}{c^2} \vec{S} = \int d^3x \, \frac{1}{4\pi c} \vec{E} \times \vec{B}$

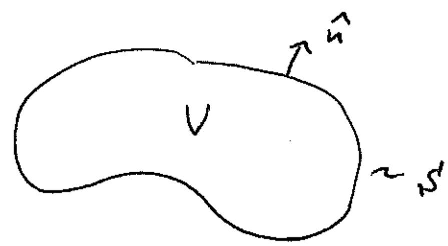
$$T^{ij}_{EM} = -\frac{1}{4\pi}\left[E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(E^2+B^2)\right]$$

Def. Maxwell stress tensor $\sigma^{ij} \equiv -T^{ij}_{EM}$.

Momentum conservation:

$$\frac{d}{dt}\left(P^i_{field} + P^i_{mech}\right) = \oint_V d^3x \, \nabla^j \sigma^{ij} = \oint_S da \, n^j \sigma^{ij}$$

$\sigma^{ij} \sim$ momentum flow into the volume



in general, energy-momentum tensor is

$$T^{\mu\nu} = \left(\begin{array}{c|c} \text{energy density} & c \cdot \text{momentum density} \\ \hline c \cdot \text{momentum density} & -\text{Maxwell stress tensor} \end{array}\right)$$

Energy - Momentum tensor:

we studied: u ~ energy density

$\vec{S}$ ~ Poynting vector (energy flow)

σ_{ij} ~ Maxwell stress tensor (momentum flow)

These seemingly unrelated quantities form one tensor under Lorentz transformations, the energy - momentum tensor:

$$T^{\mu\nu} = \begin{pmatrix} u & S^1/c & S^2/c & S^3/c \\ S^1/c & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ S^2/c & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ S^3/c & -\sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{pmatrix}$$

where $\mu, \nu = 0, 1, 2, 3$

0 ~ time component; $x^0 = tc$, $x^1 = x$, $x^2 = y$, $x^3 = z$

all the above conservation laws read

$$\boxed{\frac{\partial}{\partial x^\mu} T^{\mu\nu}_{EM} = \frac{1}{c} J_\mu F^{\mu\nu}} \quad \text{(summation over } \mu\text{)}$$

$$T_\mu{}^\mu = u + T_i{}^i = u - u = 0 \sim \text{traceless}$$

T_{11}, T_{22}, T_{33} ~ radiation pressure components
(T_{ij} is the flux of the ith component of momentum density in the $\hat{j}$ direction)

Classification of Physical Quantities by Space-Time Symmetries.

A. Spatial rotations.

$i, j = 1, 2, 3$

$$X_i \to X'_i = R_{ij} X_j \quad , \quad R_{ij} \text{ – rotation matrix}$$

$\vec{X}'^2 = \vec{X}^2$ under rotation $\Rightarrow (R_{ij} X_j)(R_{ik} X_k) = X_i X_i$

$$\Rightarrow R_{ij} R_{ik} = \delta_{jk} \Rightarrow (R^{-1})_{ij} = R_{ji} \Rightarrow \boxed{RR^T = \mathbb{1} = R^T R}$$

$\Rightarrow (\det R)^2 = 1$

$\det R = +1$ ~ rotation w/o reflection ($\det R = -1$: rot + reflection)

vectors: $A_i \to A'_i = R_{ij} A_j$

tensors: $T_{i_1 i_2 \ldots i_n} \to T'_{i_1 i_2 \ldots i_n} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_n j_n} T_{j_1 j_2 \ldots j_n}$

(definition) $\quad n$ = rank of the tensor

B. Spatial Reflection (parity)

$\vec{X} \to -\vec{X}$ ~ all vectors (or polar vectors) transform like this; now consider

$$\vec{z} = \vec{x} \times \vec{y} \Rightarrow \left.\begin{array}{l} \vec{x} \to -\vec{x} \\ \vec{y} \to -\vec{y} \end{array}\right\} \Rightarrow \vec{z} \to \vec{z}$$

axial vector (pseudovector)

Inversion is also called parity $\mathbb{P}$.

$\mathbb{P}$: vector $\rightarrow$ $-$vector, axial vector $\rightarrow$ axial vector

$p = -1$ $\qquad\qquad$ $p = +1$

Tensor of rank N: $\mathbb{P}\, T_{i_1 \ldots i_N} = (-1)^N T_{i_1 \ldots i_N}$

Pseudotensor of rank N: $\mathbb{P}\, T_{i_1 \ldots i_N} = (-1)^{N+1} T_{i_1 \ldots i_N}$

[E.g. $\vec{z} = \vec{x} \times \vec{y} \Rightarrow \underbrace{z_i}_{p=1} = \varepsilon_{ijk} \underbrace{x_j}_{p=-1} \underbrace{y_k}_{p=-1} \Rightarrow \varepsilon_{ijk}$ has $p = +1$

$\Rightarrow p = (-1)^{3+1} \Rightarrow \varepsilon_{ijk}$ is pseudotensor.]

pseudoscalar anyone? $\vec{a} \cdot (\vec{b} \times \vec{c})$.

C. Time reversal: $\boxed{t \rightarrow -t}$

$\underbrace{\mathbb{T}\, \vec{x} = \vec{x}}_{\text{T-even}}$, $\vec{p} = \frac{d\vec{x}}{dt} \Rightarrow \mathbb{T}\, \vec{p} = -\vec{p}$ ~ T-odd.

Quantity	Tensor Rank	Parity	Time Reversal
$\vec{x}$	vector	-1	1
$\vec{v} = d\vec{x}/dt$	vector	-1	-1
$\vec{p}$	vector	-1	-1
$\vec{L} = \vec{x} \times \vec{p}$	1	1	-1
$\vec{F} = m\vec{a}$	1	-1	1
$\vec{N} = \vec{x} \times \vec{F}$ (torque)	1	1	1
E ~ energy	0	1	1

Quantity	Tensor Rank	Parity	Time Reversal
ρ	0	1	1
$\vec{J}\ (=\rho\vec{v})$	1	-1	-1
$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow$			
$\vec{E}, \vec{P}, \vec{D}$	1	-1	1
$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow$			
$\vec{B}, \vec{M}, \vec{H}$	1	1	-1
$\vec{S} = \vec{E}\times\vec{H}$	1	-1	-1
T_{ij}	2	1	1