Electrostatics (SI units)

Poisson and Laplace Equations

~ consider time-independent phenomena, with electric fields only

⇒ time-independent Maxwell equations for $\vec{E}(\vec{x})$ are

$$\boxed{\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0} \quad \text{and} \quad \boxed{\vec{\nabla} \times \vec{E} = 0.}$$

Since $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$ ⇒ in t-independent case

get $\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x})$ ⇒ immediately satisfies $\vec{\nabla} \times \vec{E} = 0$.

⇒ plugging this into Gauss' law get

$$\boxed{\nabla^2 \Phi = -\rho / \varepsilon_0} \quad \text{Poisson Equation}$$

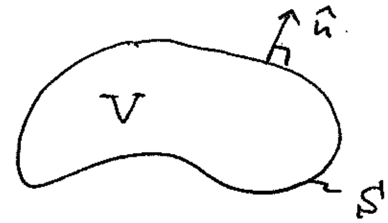
If there are no electric charges ⇒ $\rho = 0$ ⇒

get

$$\boxed{\nabla^2 \Phi = 0} \quad \text{Laplace Equation}$$

⇒ Poisson & Laplace equations are central for electrostatics

Gauss's and Coulomb's Laws



Start with Gauss's law in differential form:

$$\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0 .$$

Consider a volume V enclosed in surface area S. Here $\hat{n}$ is an outward-pointing unit vector normal to the surface.

The divergence theorem states that:

$$\boxed{\oint_S da \; \hat{n} \cdot \vec{V} = \int_V d^3x \; \vec{\nabla} \cdot \vec{V}}$$

for a vector field $\vec{V}(\vec{x})$.

Put $\vec{V} = \vec{E}$ in the divergence theorem. We get:

$$\int_V d^3x \; \underbrace{\vec{\nabla} \cdot \vec{E}}_{= \rho/\varepsilon_0} = \oint_S da \; \hat{n} \cdot \vec{E}$$

net charge in V.

$$\Rightarrow \boxed{\oint_S da \; \hat{n} \cdot \vec{E} = \frac{1}{\varepsilon_0} \int_V d^3x \, \rho(\vec{x}) = \frac{1}{\varepsilon_0} Q}$$

integral form of Gauss's Law