Last time | **Electrostatics** (cont'd)

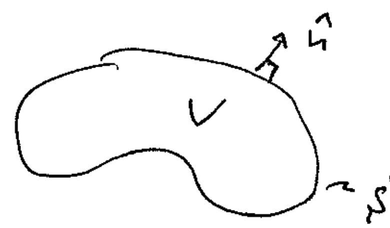
$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0} \quad \text{and} \quad \boxed{\vec{\nabla}\times\vec{E} = 0}$$

as $\vec{E} = -\vec{\nabla}\Phi \Rightarrow \boxed{\nabla^2\Phi = -\rho/\varepsilon_0}$ Poisson equation

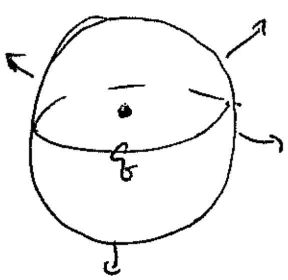
If there's no charges $\Rightarrow \rho = 0 \Rightarrow \boxed{\nabla^2\Phi = 0}$ Laplace equation

Gauss's and Coulomb's Laws

$$\oint_S da\, \hat{n}\cdot\vec{E} = \frac{Q}{\varepsilon_0}$$

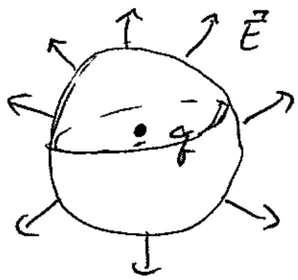


Integral form of Gauss's Law



$$\Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{x}}{|\vec{x}|^3}$$ ~ Coulomb's Law

Consider a point charge q:



applying Gauss's law we get $\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\vec{x}}{|\vec{x}|^3}}$, if the charge q is at the origin. This is <u>Coulomb's law</u>.

For many charges $q_1, \ldots, q_n$ at $\vec{x}_1, \ldots, \vec{x}_n$ we have

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3}.$$

Generalizing this to a continuous charge density $\rho(\vec{x})$ we write:

$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}}$$

generalized Coulomb's law

The Lorentz force in the static case is

$\boxed{\vec{F} = q\vec{E}}$ $\Rightarrow$ the force on charge q_1 due to charge q_2 is

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}.$$

$\Rightarrow$ if $\rho(\vec{x}) = \sum_{i=1}^{n} q_i \, \delta^3(\vec{x} - \vec{x}_i)$ $\Rightarrow$ can get back to $\vec{E}$ as a sum over i from $\int d^3x \ldots$

Note that $\dfrac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\vec{\nabla}_x \dfrac{1}{|\vec{x}-\vec{x}'|}$ $\Rightarrow$

$$\vec{E}(\vec{x}) = -\vec{\nabla}_x \left[\frac{1}{4\pi\varepsilon_0} \int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \right]$$

at the same time $\vec{E}(\vec{x}) = -\vec{\nabla}\,\Phi(\vec{x})$

$$\Rightarrow \quad \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

solution of Poisson eq'n in empty space

If this relation satisfies

$$\nabla^2 \Phi = -\rho/\varepsilon_0$$

$\Rightarrow$ this would work only if

$$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\, \delta^3(\vec{x}-\vec{x}').$$

$\Rightarrow$ let's check that this is indeed true by a direct calculation.

We need to calculate $\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|}$. To do this let's introduce a regulator ε:

$$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = \lim_{\varepsilon \to 0} \nabla^2 \frac{1}{\sqrt{(\vec{x}-\vec{x}')^2+\varepsilon^2}} = \Big|$$ suppressing the "lim"

$$= \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2}} =$$

$$= \frac{-3}{\left[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2\right]^{3/2}} + 3\,\frac{(x-x')^2+(y-y')^2+(z-z')^2}{\left[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2\right]^{5/2}}$$

$$= -3\,\frac{\varepsilon^2}{\left[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2\right]^{5/2}} \underset{\varepsilon\to 0}{=\!=} \begin{cases} 0, & |\vec{x}-\vec{x}'|^2 \neq 0 \\ \infty, & \vec{x}=\vec{x}'. \end{cases}$$

$\Rightarrow$ the function satisfies condition (i) for delta functions

$\Rightarrow$ to check (ii) we calculate

$$\int d^3x \, \frac{-3\,\varepsilon^2}{\left[\vec{x}^2+\varepsilon^2\right]^{5/2}} = \Big| \text{spherical coordinates} \Big| =$$

$$= -3\,\varepsilon^2 \cdot 4\pi \int_0^\infty dr\, \frac{r^2}{\left[r^2+\varepsilon^2\right]^{5/2}} = \Big| \tilde{r} = \frac{r}{\varepsilon} \Big| = -12\pi \underbrace{\int_0^\infty d\tilde{r}\, \frac{\tilde{r}^2}{\left[\tilde{r}^2+1\right]^{5/2}}}_{1/3}$$

$$\Rightarrow \int d^3x \, \frac{-3\varepsilon^2}{[\vec{x}^2+\varepsilon^2]^{5/2}} = -4\pi$$

$$\Rightarrow \lim_{\varepsilon \to 0} \frac{-3 \quad \varepsilon^2}{[\vec{x}^2+\varepsilon^2]^{5/2}} = -4\pi\, \delta^3(\vec{x})$$

$$\Rightarrow \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\, \delta^3(\vec{x}-\vec{x}')$$

$$\Rightarrow \nabla^2 \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \rho(\vec{x}') \, \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} =$$

$$= -\frac{1}{\varepsilon_0} \int d^3x' \rho(\vec{x}') \, \delta^3(\vec{x}-\vec{x}') = -\frac{1}{\varepsilon_0} \rho(\vec{x})$$

$\Rightarrow$ Poisson equation is satisfied!

An easier trick: $\int d^3x \; \nabla^2 \frac{1}{|\vec{x}|} = \int d^3x \; \vec{\nabla} \cdot \vec{\nabla} \frac{1}{|\vec{x}|} =$

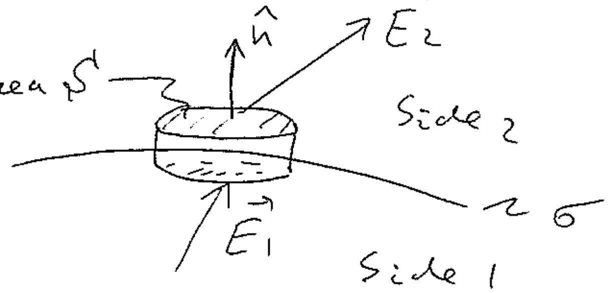
integrate over a sphere of Radius R centered at $\vec{O}$:

$$= (\text{divergence theorem}) = \oint_S \left(\vec{\nabla} \frac{1}{|\vec{x}|}\right) \cdot \hat{n} \, da = -\int R^2 d\Omega \cdot \frac{1}{R^2} = -4\pi$$

Application: Discontinuity of Electric Field at a Surface

we derived $\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$ & $\vec{\nabla} \times \vec{E} = 0$

(1)



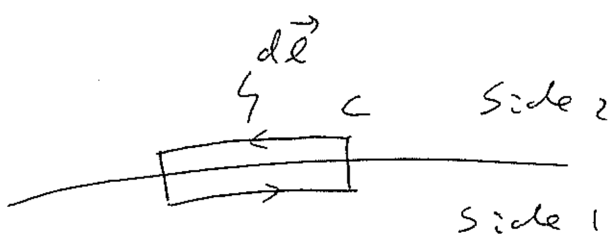
use Gauss's law:

$$\left(\vec{E}_2 \cdot \hat{n} - \vec{E}_1 \cdot \hat{n}\right) S = \frac{1}{\varepsilon_0} \sigma \cdot S$$

$$\Rightarrow \boxed{\left(\vec{E}_2 - \vec{E}_1\right) \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}}$$

~ normal component has a discontinuity if surface charge $\sigma \neq 0$.

(2)



use $\vec{\nabla} \times \vec{E} = 0$, or, equivalently,

$$\oint_c d\vec{l} \cdot \vec{E} = 0$$

$$\Rightarrow \vec{E}_2 \cdot d\vec{l} - \vec{E}_1 \cdot d\vec{l} = 0 \Rightarrow \boxed{E_{2t} = E_{1t}}$$

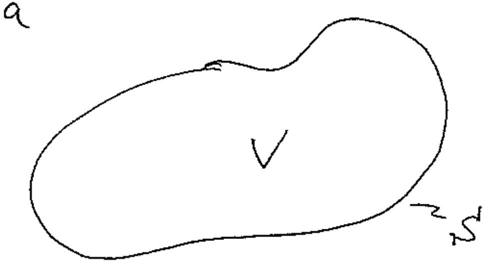
tangential component is continuous even for $\sigma \neq 0$.

Green's Theorem.

(need to solve Poisson equation with boundary conditions)

Start from divergence theorem:

$$\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \oint_S \vec{A}\cdot\hat{n}\, da$$



Put $\vec{A} = \phi\vec{\nabla}\psi$, with ϕ, ψ two arbitrary scalar fields:

$$\int_V \vec{\nabla}\cdot(\phi\vec{\nabla}\psi)\, d^3x = \oint_S \phi\, \hat{n}\cdot\vec{\nabla}\psi\, da$$

As $\vec{\nabla}\cdot(\phi\vec{\nabla}\psi) = \phi\nabla^2\psi + (\vec{\nabla}\phi)(\vec{\nabla}\psi)$

and denoting $\hat{n}\cdot\vec{\nabla}\psi = \frac{\partial\psi}{\partial n}$ we get

$$\int_V d^3x \left[\phi\nabla^2\psi + (\vec{\nabla}\phi)(\vec{\nabla}\psi)\right] = \oint_S \phi\frac{\partial\psi}{\partial n}\, da$$

Green's first identity.

Swap $\phi \leftrightarrow \psi$:

$$\int_V d^3x \left[\psi\nabla^2\phi + (\vec{\nabla}\phi)(\vec{\nabla}\psi)\right] = \oint_S \psi\frac{\partial\phi}{\partial n}\, da$$

& subtract from the 1st identity:

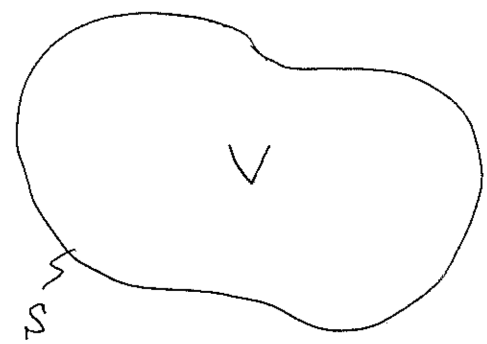
$$\int_V d^3x \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi \right] = \oint_S \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] da$$

Green's second identity or Green's theorem.

Solution of Poisson Equation:

Dirichlet & Neumann Boundary Conditions

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \quad \text{in volume } V:$$



(I) ϕ is specified on S

~ Dirichlet boundary condition

(II) $\dfrac{\partial \phi}{\partial n}$ is specified on S

~ Neumann boundary condition

Uniqueness of the solution:

suppose there are 2 solutions ~ ϕ_1 & ϕ_2

$$\nabla^2 \phi_1 = -\frac{\rho}{\varepsilon_0} \quad \& \quad \nabla^2 \phi_2 = -\frac{\rho}{\varepsilon_0} \implies \text{define}$$

$u = \phi_1 - \phi_2 \implies \nabla^2 u = 0 \implies$ put $\phi = \psi = u$ in first Green's identity

$$\int_V d^3x \left[u \underbrace{\nabla^2 u}_{=0} + |\vec{\nabla} u|^2 \right] = \oint_S u \frac{\partial u}{\partial n} da$$

$$\Rightarrow \int_V d^3x \, |\vec{\nabla} u|^2 = 0$$

as for Dirichlet $u = 0$ on S

for Neumann $\frac{\partial u}{\partial n} = 0$ on S

$\vec{\nabla} u = 0 \Rightarrow u = \text{const}$ (b.c.)

$\Rightarrow u = 0$ D. in V $\Rightarrow$

$\Rightarrow$ solution is unique, $\phi_1 = \phi_2$.

(in case of Neumann one may have $u = \text{const}$ ~ not important, ϕ is defined up to a constant anyway)

Green Functions

(Green had no formal math education when he published it all in 1828 at the age of ≈35)

Suppose you have a linear differential equation

$$\hat{L}_x \, \psi(x) = J(x)$$

where $J(x)$ is known, $\hat{L}_x$ is some (linear) differential operator and $\psi(x)$ is to be found.

(If we know the Green function of operator $\hat{L}_x$ <u>defined</u> by $\hat{L}_x \, G(\vec{x}, \vec{x}') = \delta^3(\vec{x} - \vec{x}')$, then

$$\psi(x) = \int d^3x' \; J(\vec{x}') \cdot G(\vec{x}, \vec{x}')$$ would be

the solution of our equation. Works for any linear operator $\hat{L}_x$, and any "good" function $J(x)$.

In our case, define Green function by

$$\left\{ \nabla^2 G(\vec{x}, \vec{x}') = -4\pi \, \delta^3(\vec{x} - \vec{x}') \right\}$$

We know that $$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}')$$

for any F such that $\nabla^2 F(\vec{x}, \vec{x}') = 0$ <u>in V</u>.

Substitute $\phi(\vec{x}') = \Phi(\vec{x}')$ the potential and

$\psi(\vec{x}') = G(\vec{x}', \vec{x})$ into the second Green's identity:

$$\int_V d^3x' \left[\Phi(\vec{x}') \overbrace{\nabla'^2 G(\vec{x}', \vec{x})}^{-4\pi\,\delta^3(\vec{x} - \vec{x}')} - G(\vec{x}', \vec{x}) \overbrace{\nabla'^2 \Phi}^{-\rho/\varepsilon_0} \right] =$$

$$= \oint_S \left[\Phi(\vec{x}') \frac{\partial G(\vec{x}', \vec{x})}{\partial n'} - G(\vec{x}', \vec{x}) \frac{\partial \Phi}{\partial n'} \right] da'$$

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \; G(\vec{x}', \vec{x}) \, \rho(\vec{x}') + \frac{1}{4\pi} \oint_S \left[G(\vec{x}', \vec{x}) \frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}', \vec{x})}{\partial n'} \right] da'}$$

"master formula"!

Zangwill, pp. 250-252 (Sec. 8.4