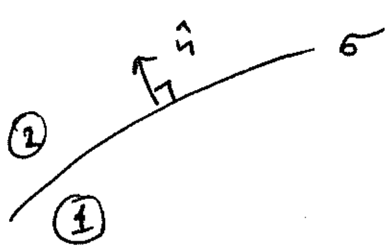
Last time Solution of Poisson eq'n in infinite space:

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

Solves $\nabla^2 \Phi = -\rho/\varepsilon_0$ if

$$\boxed{\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi \, \delta^3(\vec{x}-\vec{x}')}$$

Boundary matchings:



$$\boxed{\left(\vec{E}_2 - \vec{E}_1\right)\cdot\hat{n} = \sigma/\varepsilon_0}$$

$$\boxed{E_{2t} = E_{1t}}$$

Proved the following identities for general functions

$\psi(\vec{x})$ and $\phi(\vec{x})$:

$$\boxed{\int_V d^3x \left[\phi \nabla^2\psi + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\psi)\right] = \oint_{S} da \, \phi \frac{\partial\psi}{\partial n}}$$

Green's 1st identity

$$\boxed{\int_V d^3x \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] = \oint_S da\left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right]}$$

Green's 2nd identity aka Green's theorem

Solution of Poisson Equation: Dirichlet & Neumann Boundary Conditions (cont'd)

$$\nabla^2 \Phi = -\rho/\varepsilon_0$$

(I) Dirichlet: Φ specified on S.

(II) Neumann: $\frac{\partial \Phi}{\partial n}$ given on S.

We proved uniqueness of the solution with Dirichlet and Neumann boundary conditions.

(For Neumann case, solution is unique up to a constant.)

$$\int_V d^3x \left[u \underbrace{\nabla^2 u}_{=0} + |\vec{\nabla} u|^2 \right] = \oint_S u \frac{\partial u}{\partial n}\, da$$

$$\Rightarrow \quad \int_V d^3x \, |\vec{\nabla} u|^2 = 0$$

as for Dirichlet $u = 0$ on S

for Neumann $\frac{\partial u}{\partial n} = 0$ on S

$\vec{\nabla} u = 0 \Rightarrow u = \text{const}$

(b.c.)

$\Rightarrow u = 0$ D.

in V $\Rightarrow$

$\Rightarrow$ solution is unique, $\phi_1 = \phi_2$.

(in case of Neumann one may have $u = \text{const}$ ~ not important, ϕ is defined up to a constant anyway)

Green Functions

(Green had no formal math education when he published it all in 1828 at the age of ~35)

~see p. 251 in Zangwill

Suppose you have a linear differential equation

$$\hat{L}_x \, \Psi(\vec{x}) = J(\vec{x})$$

where $J(\vec{x})$ is known, $\hat{L}_x$ is some (linear) differential operator and $\Psi(x)$ is to be found.

If we know the Green function of operator $\hat{L}_x$ defined by $\quad \hat{L}_x \, G(\vec{x}, \vec{x}\,') = \delta^3(\vec{x} - \vec{x}\,')$, then

$$\psi(x) = \int d^3x' \; J(\vec{x}') \cdot G(\vec{x}, \vec{x}')$$ would be the solution of our equation. Works for any linear operator $\hat{L}_x$, and any "good" function $J(x)$.

In our case, define Green function by

$$\left\{ \nabla^2 G(\vec{x}, \vec{x}') = -4\pi \, \delta^3(\vec{x} - \vec{x}') \right\}$$

We know that $G(\vec{x}, \vec{x}') = \dfrac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}')$

for any F such that $\nabla^2 F(\vec{x}, \vec{x}') = 0$ in V.

Substitute $\phi(\vec{x}') = \Phi(\vec{x}')$ the potential and $\psi(\vec{x}') = G(\vec{x}', \vec{x})$ into the second Green's identity:

$$\int_V d^3x' \Big[\Phi(\vec{x}') \underbrace{\nabla'^2 G(\vec{x}', \vec{x})}_{-4\pi \delta^3(\vec{x} - \vec{x}')} - G(\vec{x}', \vec{x}) \underbrace{\nabla'^2 \Phi}_{-\rho/\varepsilon_0} \Big] =$$

$$= \oint_S \left[\Phi(\vec{x}') \frac{\partial G(\vec{x}', \vec{x})}{\partial n'} - G(\vec{x}', \vec{x}) \frac{\partial \Phi}{\partial n'} \right] da'$$

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \; G(\vec{x}', \vec{x}) \, \rho(\vec{x}') + \frac{1}{4\pi} \oint_S \left[G(\vec{x}', \vec{x}) \frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}', \vec{x})}{\partial n'} \right] da'}$$

"master formula".

Zangwill, pp. 250-252 (Sec. 8.4,

Most of the Green functions we'll encounter will be symmetric:

$$G(\vec{x}, \vec{x}\,') = G(\vec{x}\,', \vec{x})$$

=> for those the "master formula" becomes:

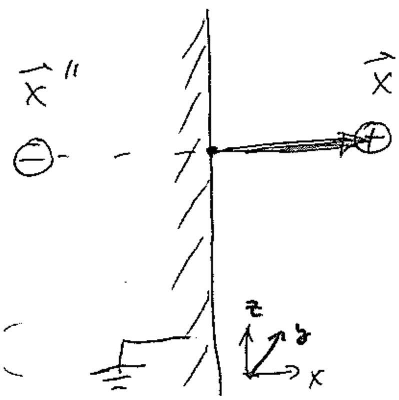
$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \, G(\vec{x}, \vec{x}\,') \, \rho(\vec{x}\,') + \frac{1}{4\pi} \oint_S da' \cdot \left[G(\vec{x}, \vec{x}\,') \frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}\,') \frac{\partial G(\vec{x}, \vec{x}\,')}{\partial n'} \right]}$$

cf. Jackson Sec. 1.10
pp. 38-40

Use the freedom of redefining $G \to G + F$, where $\nabla^2 F = 0$, to fix boundary conditions for $G(\vec{x}, \vec{x}')$.

Example: conductors are equipotential (if not $\Rightarrow$ get $\vec{E} \neq 0$ $\Rightarrow$ will become equipotential)

$\Rightarrow$ natural candidate for Dirichlet boundary conditions ~ conducting surfaces as boundaries



infinite flat conductor

$\Rightarrow$ interested in potential outside conductor

$$G = \frac{1}{|\vec{x} - \vec{x}'|} \quad \text{outside}$$

Can add $F = -\dfrac{1}{|\vec{x} - \vec{x}''|}$ with $(-x', y', z') = \vec{x}''$ as

$\nabla^2 F = 0$ in the volume of interest.

One gets $G' = \dfrac{1}{|\vec{x} - \vec{x}'|} - \dfrac{1}{|\vec{x} - \vec{x}''|}$ $\Rightarrow$ $G' = 0$ on the surface

---

(I) To solve Dirichlet b.c. problem choose

$G_D(\vec{x}, \vec{x}') = 0$ for $\vec{x}$ on S $\Rightarrow$ using master formula

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, G_D(\vec{x}, \vec{x}') \rho(\vec{x}') - \frac{1}{4\pi} \oint_S \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'} \, da'}$$

$\Rightarrow$ if one knows boundary condition $\phi(\vec{x})$ on S and $G_D(\vec{x}, \vec{x}')$, along with the charge density $\rho(\vec{x}) \Rightarrow$ can find $\phi(\vec{x})$ anywhere in V.

II to solve Neumann boundary conditions:

can't just put $\frac{\partial G_N(\vec{x}, \vec{x}')}{\partial n'} = 0$, as due to

$$\nabla^2 G_N(\vec{x}, \vec{x}') = -4\pi\, \delta(\vec{x} - \vec{x}')$$

$$\Rightarrow \int_V d^3\vec{x}'\, \nabla'^2 G_N(\vec{x}, \vec{x}') = \int_V d^3x'\, \vec{\nabla}' \cdot \vec{\nabla}' G_N(\vec{x}, \vec{x}') =$$

$$\begin{pmatrix}\text{divergence} \\ \text{thm}\end{pmatrix} = \oint_S da'\, \frac{\partial G_N(\vec{x}, \vec{x}')}{\partial n'} \underset{\substack{\uparrow \\ \text{if } \vec{x} \in V}}{=} -4\pi \quad \text{due to def. of } G_N$$

$$\Rightarrow \frac{\partial G_N}{\partial n'} = 0 \text{ does not work}$$

Instead: $$\frac{\partial G_N}{\partial n'} = \frac{-4\pi}{\text{area of } S} = -\frac{4\pi}{S}$$

$$\Rightarrow \boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\, G_N(\vec{x}, \vec{x}')\, \rho(\vec{x}') + \frac{1}{4\pi}\oint_S G_N(\vec{x}, \vec{x}')\frac{\partial \Phi}{\partial n'}\, da' + \langle \Phi \rangle_{\text{surface}}}$$

where $\langle \Phi \rangle_{\text{surface}} \equiv \frac{1}{S}\oint_S \Phi(\vec{x}')\, da'$. [usually S is infinite. $\Rightarrow$ can drop. does not affect $\vec{E}$.]