Last time

Solution of Poisson Equation (cont'd)

Green function of the Laplace operator:

$$\boxed{\nabla_x^2 \, G(\vec{x}, \vec{x}\,') = -4\pi \, \delta^3(\vec{x}, \vec{x}\,')}$$

Using Green's theorem, we obtained the "master formula":

$$\boxed{\begin{aligned}\Phi(\vec{x}) &= \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \, G(\vec{x}, \vec{x}\,') \rho(\vec{x}\,') \\ &+ \frac{1}{4\pi} \oint_S da' \left[G(\vec{x}, \vec{x}\,') \frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}\,') \frac{\partial G(\vec{x}, \vec{x}\,')}{\partial n'} \right]\end{aligned}}$$

(I) Dirichlet boundary conditions:

require $G_D(\vec{x}, \vec{x}\,') = 0$ for $\vec{x}\,'$ on S

$$\Rightarrow \boxed{\begin{aligned}\Phi(\vec{x}) &= \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \, G_D(\vec{x}, \vec{x}\,') \rho(\vec{x}\,') \\ &- \frac{1}{4\pi} \oint_S da' \, \Phi(\vec{x}\,') \frac{\partial G_D(\vec{x}, \vec{x}\,')}{\partial n'}\end{aligned}}$$

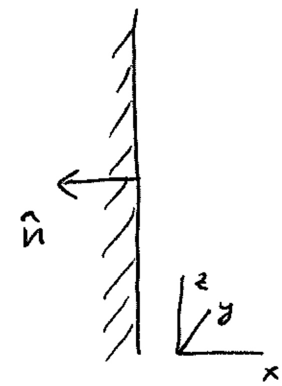
(II) Neumann boundary conditions: put $\dfrac{\partial G_N(\vec{x}, \vec{x}\,')}{\partial n'} = 0$

for $\vec{x}\,'$ on S'

$$\Rightarrow \boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \, G_N(\vec{x}, \vec{x}\,') \rho(\vec{x}\,') + \frac{1}{4\pi} \oint_S da' \, G_N(\vec{x}, \vec{x}\,') \frac{\partial \Phi}{\partial n'} + \langle \Phi \rangle_{\text{surface}}}$$

Know ρ, G_N and $\left.\frac{\partial \Phi}{\partial n}\right|_S \Rightarrow$ get $\Phi(\vec{x})$ everywhere inside V.

Example | Dirichlet problem in semi-infinite space with $\rho = 0$:



$$G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}''|}, \quad \vec{x}'' = (-x', y', z').$$

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \oint_S da'\, \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}$$

$\uparrow$ for $\rho = 0$

$$\Rightarrow \text{need } \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'} = -\frac{\partial G_D(\vec{x}, \vec{x}')}{\partial x'} = \frac{x' - x}{|\vec{x} - \vec{x}'|^3} - \frac{x' + x}{|\vec{x} - \vec{x}''|^3}$$

$$\Rightarrow \left.\frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}\right|_{x'=0} = \frac{-2x}{|y'\hat{y} + z'\hat{z} - \vec{x}|^3}$$

Boundary conditions: $\Phi(x, y, z) = \begin{cases} \Phi_0, & y^2 + z^2 < a^2 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow \Phi(\vec{x}) = -\frac{1}{4\pi} \Phi_0 \int dy'\, dz' \frac{-2x}{|y'\hat{y} + z'\hat{z} - \vec{x}|^3}$$

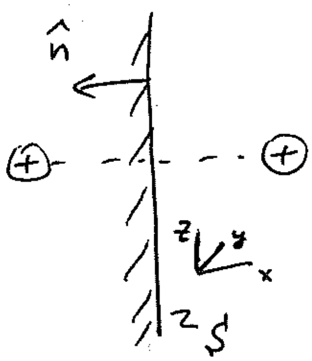
Suppose $y = z = 0$ (Φ along the x-axis):

$$\Phi(x, 0, 0) = \frac{\Phi_0}{2\pi} x \int_0^a d\rho \cdot \rho \cdot 2\pi \cdot \frac{1}{[\rho^2 + x^2]^{3/2}} \Rightarrow$$

$$\Rightarrow \boxed{\Phi(x,0,0) = \Phi_0 \left[1 - \frac{x}{\sqrt{x^2+a^2}} \right]}$$

Check: $\Phi(0,0,0) = \Phi$, boundary condition is satisfied

Example Same problem with Neumann boundary conditions: $\frac{\partial G_N}{\partial n'} = -\frac{4\pi}{S} = 0$ as $S' = \infty$ $\Rightarrow$ like zero $\vec{E}$.



To find $G_N(\vec{x}, \vec{x}')$ use anti-image:

$$\boxed{G_N(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \frac{1}{|\vec{x}-\vec{x}''|}}$$

Check: $$\left.\frac{\partial G_N}{\partial n'}\right|_{x'=0} = -\left.\frac{\partial G_N}{\partial x'}\right|_{x'=0} = \left.\left(\frac{x'-x}{|\vec{x}-\vec{x}'|^3} + \frac{x'+x}{|\vec{x}-\vec{x}''|^3} \right)\right|_{x'=0} = 0.$$

Boundary condition: $\frac{\partial \Phi}{\partial n} = -\frac{\partial \Phi}{\partial x} = E_0$ ~ constant $\vec{E}$-field

Assume that $\rho = 0 \Rightarrow$ no charges. $\Rightarrow$ using master f-la

$$\Phi(\vec{x}) = \frac{1}{4\pi} \oint_S da' \, E_0 \cdot \left.\left(\frac{1}{|\vec{x}-\vec{x}'|} + \frac{1}{|\vec{x}-\underbrace{\vec{x}''}_{(-x',y',z')}|} \right)\right|_{x'=0} + \langle \Phi \rangle_{surface} = \frac{E_0}{4\pi} \oint_S da' \cdot$$

$$\cdot \left.\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}} \right]\right|_{x'=0} + \langle \Phi \rangle_{surface} =$$

$$= \frac{E_0}{\cancel{2\pi}} \int_0^\infty d\rho \cdot \rho \cdot \cancel{2\pi} \frac{1}{\sqrt{x^2+\rho^2}} = E_0 \left. (x^2+\rho^2) \right|_0^\infty = -E_0 x + \text{const} + \langle \Phi_{surface} \rangle$$

$$\Rightarrow \phi(\vec{x}) = -E_0 X$$

$\Rightarrow$ constant $\underset{\text{uniform}}{\wedge}$ $\vec{E}$ - field everywhere $\overset{\text{in } V}{\vee}$, $\vec{E} = -\vec{\nabla}\Phi = \hat{x} E_0$

Electrostatic Energy

$$u = \frac{E^2 + B^2}{8\pi} \underset{B=0}{\Rightarrow} u = \frac{E^2}{8\pi}$$ in electrostatics (Gaussian units).

Net energy is $\mathcal{E} = \int d^3x \, u$

In SI $u = \frac{\varepsilon_0}{2} E^2 \Rightarrow \boxed{\mathcal{E} = \int d^3x \, \frac{\varepsilon_0}{2} \vec{E}^2}$

$$\vec{E} = -\vec{\nabla}\Phi \Rightarrow \mathcal{E} = \frac{\varepsilon_0}{2}\int d^3x \, (\vec{\nabla}\Phi)^2 = \frac{\varepsilon_0}{2}\int d^3x \, \cdot$$

$$\cdot \Big[\underbrace{\vec{\nabla}(\Phi \vec{\nabla}\Phi)}_{\text{divergence th'm}} - \Phi \underbrace{\nabla^2\Phi}_{-\rho/\varepsilon_0 \text{ (Poisson eq'n)}} \Big] = \frac{1}{2}\int d^3x \, \Phi(x)\rho(x)$$

$\Rightarrow$ surface integral $\Rightarrow$ drop

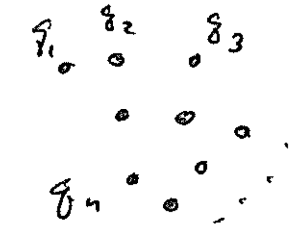
$$\Rightarrow \boxed{\mathcal{E} = \frac{1}{2}\int d^3x \, \Phi(\vec{x})\rho(\vec{x})}$$

Since $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$ (unlimited space)

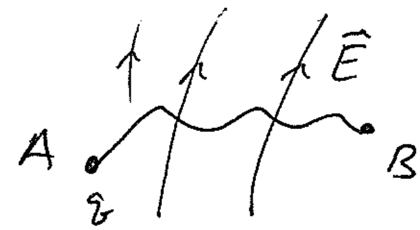
$$\Rightarrow \boxed{\mathcal{E} = \frac{1}{8\pi\varepsilon_0}\int d^3x\, d^3x' \, \frac{\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

For a discrete set of charges this becomes

$$\boxed{\mathcal{E} = \frac{1}{8\pi\varepsilon_0}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{q_i\, q_j}{|\vec{x}_i - \vec{x}_j|}}$$



Work needed to move a charge in $\vec{E}$-field:

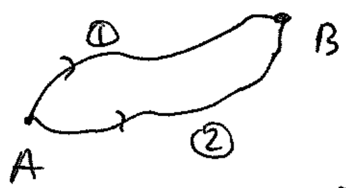


$$W = -\int_A^B d\vec{\ell}\cdot\vec{F} = -q\int_A^B d\vec{\ell}\cdot\vec{E} =$$

$$= q\int_A^B d\vec{\ell}\cdot\vec{\nabla}\Phi = q\int_A^B d\Phi = q(\Phi_B - \Phi_A)$$

$\Rightarrow$ work is independent of path!

Consider two distinct paths:



$$W_1 - W_2 = -q\int_{①} d\vec{\ell}\cdot\vec{E} + q\int_{②} d\vec{\ell}\cdot\vec{E} =$$

$$= q\oint_{C=②-①} d\vec{\ell}\cdot\vec{E} \underset{\text{Stokes's th'm}}{=} q\oint_S da\; \hat{n}\cdot(\underbrace{\vec{\nabla}\times\vec{E}}_{=0}) = 0$$

$$\Rightarrow W_1 = W_2$$

Stokes's theorem:

$\vec{V}$ = vector field

$$\boxed{\oint_C d\vec{\ell}\cdot\vec{V} = \int_S da\; \hat{n}\cdot(\vec{\nabla}\times\vec{V})}$$

 C ~ contour, S ~ enclosed area (not necessarily planar)