$$\Rightarrow \phi(\vec{x}) = -E_0 x$$

$\Rightarrow$ constant $\vec{E}$-field everywhere in V (uniform), $\vec{E} = -\vec{\nabla}\Phi = \hat{x} E_0$

Electrostatic Energy

$$u = \frac{E^2 + B^2}{8\pi} \underset{B=0}{\Rightarrow} u = \frac{E^2}{8\pi}$$ in electrostatics (Gaussian units).

Net energy is $\mathcal{E} = \int d^3x \, u$

In SI $u = \frac{\varepsilon_0}{2} E^2 \Rightarrow \boxed{\mathcal{E} = \int d^3x \, \frac{\varepsilon_0}{2} \vec{E}^2}$

$$\vec{E} = -\vec{\nabla}\Phi \Rightarrow \mathcal{E} = \frac{\varepsilon_0}{2}\int d^3x \left(\vec{\nabla}\Phi\right)^2 = \frac{\varepsilon_0}{2}\int d^3x \cdot$$

$$\cdot \Big[\underbrace{\vec{\nabla}\left(\Phi \vec{\nabla}\Phi\right)}_{\text{divergence th'm}} - \Phi \underbrace{\nabla^2 \Phi}_{-\rho/\varepsilon_0 \text{ (Poisson eq'n)}} \Big] = \frac{1}{2}\int d^3x \, \Phi(x)\rho(x)$$

$\Rightarrow$ surface integral $\Rightarrow$ drop

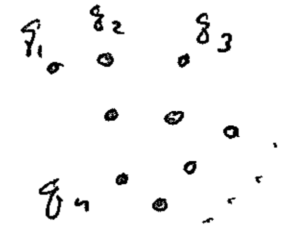
$$\Rightarrow \boxed{\mathcal{E} = \frac{1}{2}\int d^3x \, \Phi(\vec{x})\rho(\vec{x})}$$

Since $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$ (unlimited space)

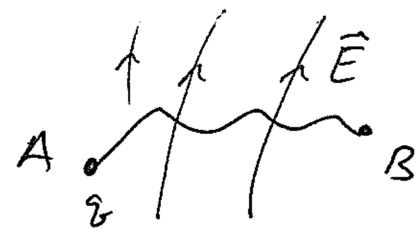
$$\Rightarrow \boxed{\mathcal{E} = \frac{1}{8\pi\varepsilon_0}\int d^3x\, d^3x' \, \frac{\rho(\vec{x})\,\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}}$$

For a discrete set of charges this becomes

$$\boxed{\mathcal{E} = \frac{1}{8\pi\varepsilon_0}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{q_i\, q_j}{|\vec{x}_i - \vec{x}_j|}}$$



Work needed to move a charge in $\vec{E}$-field:

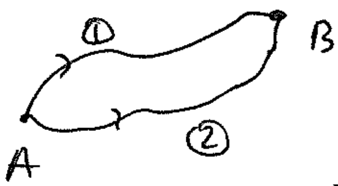


$$W = -\int_A^B d\vec{\ell}\cdot\vec{F} = -q\int_A^B d\vec{\ell}\cdot\vec{E} =$$

$$= q\int_A^B d\vec{\ell}\cdot\vec{\nabla}\Phi = q\int_A^B d\Phi = q(\Phi_B - \Phi_A)$$

$\Rightarrow$ work is independent of path!

Consider two distinct paths:



$$W_1 - W_2 = -q\int_{①} d\vec{\ell}\cdot\vec{E} + q\int_{②} d\vec{\ell}\cdot\vec{E} =$$

$$= q\oint_{C=②-①} d\vec{\ell}\cdot\vec{E} \underset{\text{Stokes's th'm}}{=} q\oint_S da\, \hat{n}\cdot\underbrace{(\vec{\nabla}\times\vec{E})}_{=0} = 0$$

$\Rightarrow W_1 = W_2$

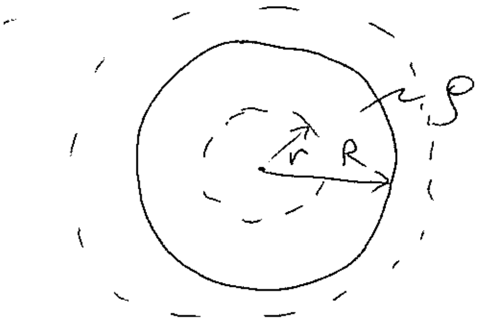
<u>Stokes's theorem:</u>

$$\boxed{\oint_C d\vec{\ell}\cdot\vec{V} = \int_S da\, \hat{n}\cdot(\vec{\nabla}\times\vec{V})}$$

$\vec{V}$ = vector field

C = contour, S = enclosed area (not necessarily planar)

Let's find the energy of a uniformly charged sphere:



(I) Inside the sphere

$$4\pi r^2 E_{in} = \frac{1}{\varepsilon_0} \frac{4}{3}\pi r^3 \rho$$

$$\Rightarrow \quad \boxed{E_{in} = \frac{1}{3\varepsilon_0} r\rho}$$

II Outside the sphere: $4\pi r^2 E_{out} = \frac{1}{\varepsilon_0}\frac{4}{3}\pi R^3 \rho$

$$\Rightarrow \quad \boxed{E_{out} = \frac{1}{3\varepsilon_0}\frac{R^3}{r^2}\rho}$$

$$\mathcal{E} = \frac{\varepsilon_0}{2}\int d^3x\, \vec{E}^2 = \frac{\varepsilon_0}{2}\int dr\, r^2 \overbrace{\sin\theta\, d\theta\, d\varphi}^{4\pi}\, \vec{E}^2 =$$

$$= 2\pi\varepsilon_0 \left\{ \int_0^R dr\, r^2 E_{in}^2 + \int_R^\infty dr\, r^2 E_{out}^2 \right\} =$$

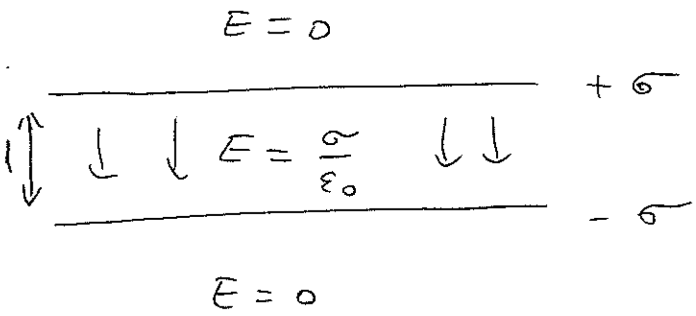
$$= 2\pi\varepsilon_0 \left\{ \int_0^R dr\, r^4 \frac{\rho^2}{9\varepsilon_0^2} + \int_R^\infty dr\, \frac{1}{r^2}\frac{R^6\rho^2}{9\varepsilon_0^2} \right\} =$$

$$= \frac{2\pi}{9}\frac{\rho^2}{\varepsilon_0}\left\{ \frac{R^5}{5} + R^5 \right\} = \frac{4\pi}{15}\frac{\rho^2}{\varepsilon_0} R^5$$

Defining $Q = \rho \cdot \frac{4}{3}\pi R^3 \Rightarrow \mathcal{E} = \frac{\cancel{4\pi}}{15}\frac{1}{\varepsilon_0 R}\frac{9Q^2}{(4\pi)^{\cancel{2}}} \Rightarrow$

$$\Rightarrow \quad \boxed{\mathcal{E} = \frac{3}{20\pi}\frac{Q^2}{\varepsilon_0 R}}.$$

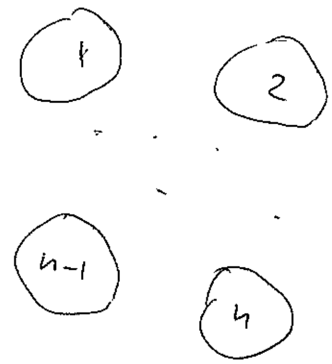
Capacitance



$$\Rightarrow w = \frac{\varepsilon_0}{2} E^2 = \frac{\sigma^2}{2\varepsilon_0}$$

General definition of capacitance:

suppose we have n conductors:



at potentials $V_1, V_2, \dots, V_n$

The energy stored in the system

is
$$\mathcal{E} = \frac{1}{2}\int d^3x\, \rho(\vec{x})\, \Phi(\vec{x}) = \frac{1}{2}\sum_{i=1}^{n} V_i \int\limits_{\substack{\text{Volume}\\ \text{of ith}\\ \text{conductor}}} \rho(\vec{x})\, d^3x =$$

$$= \frac{1}{2}\sum_{i=1}^{n} V_i Q_i$$

potentials V_i depend on Q_i linearly $\left(\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}\right)$

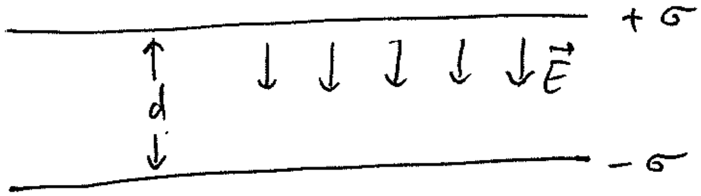
$\Rightarrow$ let's write
$$Q_i = \sum_{j=1}^{n} C_{ij} V_j$$

C_{ij}, $i \neq j$ coefficients of induction

C_{ii} ~ capacitances

$$\mathcal{E} = \frac{1}{2}\sum_{i,j=1}^{n} C_{ij} V_i V_j$$

Flat-plates capacitor:

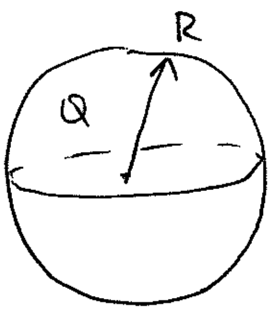


$$V = E\cdot d = \frac{\sigma}{\varepsilon_0} d = \frac{Q}{C} \quad Q = \sigma \cdot S$$

$$\Rightarrow \boxed{C = \frac{\varepsilon_0 S}{d}} \Rightarrow \boxed{\frac{C}{S} = \frac{\varepsilon_0}{d}}$$ capacitance per unit area.

$$[C] = \frac{1 \text{ Coulomb}}{1 \text{ Volt}} = 1 \text{ Farad}$$

Conducting sphere with charge Q and radius R:

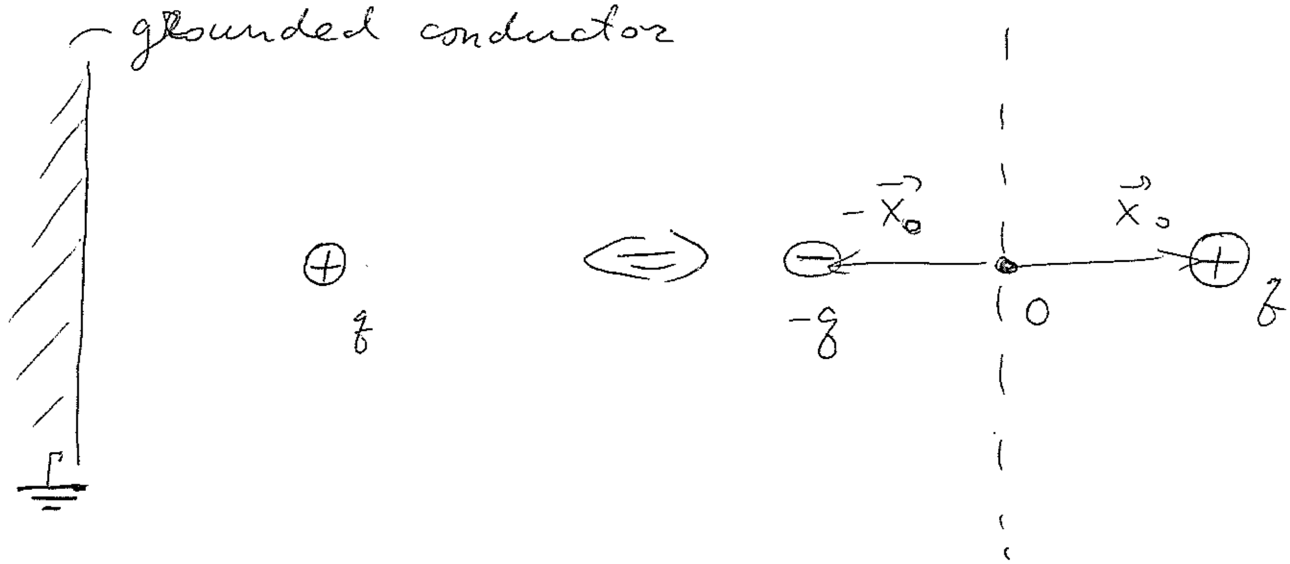


$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \Rightarrow \Phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \Rightarrow$$

$$\Rightarrow C = \frac{Q}{V(r=R)} = \frac{Q}{\frac{1}{4\pi\varepsilon_0}\frac{Q}{R}} = 4\pi\varepsilon_0 R$$

$$\Rightarrow \boxed{C = 4\pi\varepsilon_0 R}$$ capacitance of a conducting sphere.

Method of Images.



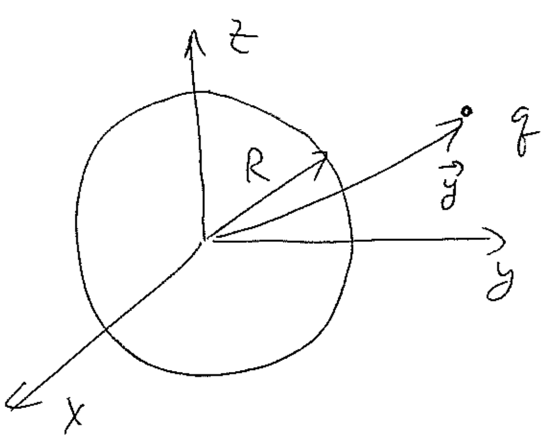
replace conductor by image charge(s):

$\phi = 0$ in conductor

$$\phi_{\text{charge}+\text{image}} = \left(\frac{1}{|\vec{x} - \vec{x}_0|} - \frac{1}{|\vec{x} + \vec{x}_0|} \right) \frac{q}{4\pi\varepsilon_0}$$

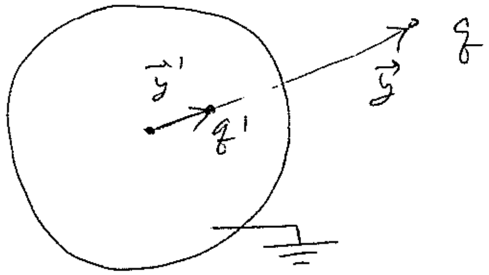
$= 0$ if $\vec{x}$ is on (what used to be) boundary.

Dirichlet Green function for a Sphere.



First let's solve the problem of a point charge q & grounded conducting sphere

↳ at $\vec{y}$



The potential of charge q and image charge q' is

$$\phi(\vec{x}) = \left(\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|} \right) \frac{1}{4\pi\varepsilon_0}$$

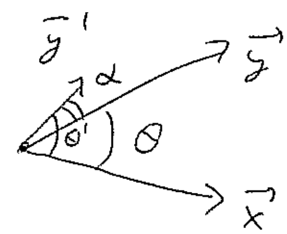
Let's demand $\phi(|\vec{x}|=R) = 0$ (grounded conductor):

$$\left.\frac{q}{|\vec{x}-\vec{y}|}\right|_{|\vec{x}|=R} = -\left.\frac{q'}{|\vec{x}-\vec{y}'|}\right|_{|\vec{x}|=R}$$

$\Rightarrow$ sign q' = $-$ sign q

Square:
(assume that $\vec{y}, \vec{y}'$ & $\vec{x}$ lie in same plane)

$$\left.\frac{q^2}{|\vec{x}-\vec{y}|^2}\right|_{|\vec{x}|=R} = \left.\frac{q'^2}{|\vec{x}-\vec{y}'|^2}\right|_{|\vec{x}|=R}$$



$$q^2\left(R^2+y'^2-2Ry'\cos\theta'\right) = q'^2\left(R^2+y^2-2Ry\cos\theta\right)$$

$\theta' = \theta + \alpha$ & identity should work for any θ

$\Rightarrow$ as $\cos\theta' = \cos(\theta+\alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$ (ahh)

$\Rightarrow$ no $\sin\theta$ on r.h.s. $\Rightarrow \sin\alpha = 0 \Rightarrow \alpha = 0, \pi; \cos\alpha = \pm 1.$

$\Rightarrow \quad q^2(R^2+y'^2) = q'^2(R^2+y^2)$ and $\pm q^2 y' = q'^2 y$.

$\uparrow$ from $\cos\alpha$.

as $y, y' > 0$ (magnitudes of vectors) $\Rightarrow \cos\alpha = 1$

$\alpha = 0.$

$$\Rightarrow \quad q^2 y' = q'^2 y \Rightarrow y' = \frac{q'^2 y}{q^2}$$

$$\Rightarrow \quad q^2 R^2 + q^2 \frac{q'^4 y^2}{q^4} = q'^2 R^2 + q'^2 y^2$$

$$q'^4 \frac{y^2}{q^2} - q'^2 (R^2 + y^2) + q^2 R^2 = 0$$

$$q'^2 = \frac{1}{2 y^2/q^2} \left[+ R^2 + y^2 \pm \sqrt{(R^2+y^2)^2 - 4y^2 R^2} \right] =$$

$$= \frac{q^2}{2y^2} \left[R^2 + y^2 \pm \underbrace{| R^2 - y^2 |}_{= y^2 - R^2 \text{ as } y > R} \right]$$

$$\Rightarrow \text{①}\ q'^2 = q^2 \quad \text{and ②}\ q'^2 = q^2 \frac{R^2}{y^2}.$$

① $\Rightarrow y' = y \Rightarrow$ put $-q$ on top of $q \Rightarrow$ get $\emptyset$;

however, $y' < R < y \Rightarrow$ can't have this case

② $\Rightarrow \boxed{q' = -q \frac{R}{y}} \quad \boxed{y' = \frac{R^2}{y}}$ Mathematical transformation of inversion.

The potential is then

$$\boxed{\Phi(\vec{x}) = \left(\frac{q}{|\vec{x} - \vec{y}|} - q \frac{R}{y} \frac{1}{|\vec{x} - \frac{R^2}{y^2}\vec{y}|} \right)} \times \frac{1}{4\pi\varepsilon_0}$$

$\Rightarrow$ Dirichlet Green function is

$$\boxed{G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{R}{x'} \frac{1}{|\vec{x} - \frac{R^2}{x'^2}\vec{x}'|}}$$

<u>Exercise</u>: find the force between charge q & conducting sphere:

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q \cdot q'}{|\vec{y} - \vec{y}'|^2} = -\frac{1}{4\pi\varepsilon_0} \frac{q^2 R/y}{\left[y - \frac{R^2}{y}\right]^2}.$$

The problem of finding $\Phi(\vec{x})$ for a charge q and any sphere is a Dirichlet problem:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \, G_D(\vec{x}, \vec{x}') \rho(\vec{x}') - \frac{1}{4\pi} \oint_S da' \, \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}.$$

$= -\frac{\partial G_D}{\partial r'}$

Here $\Phi = 0$ on S for grounded conducting sphere.