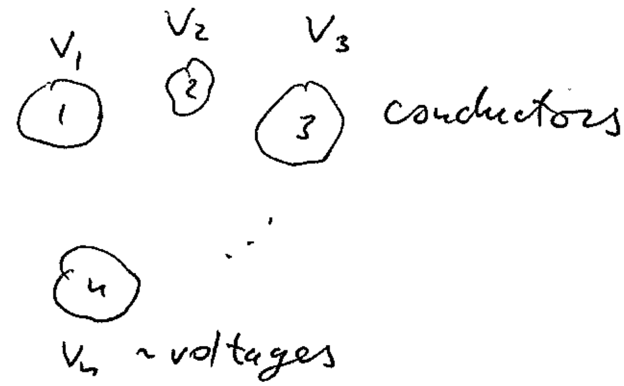
Last time | Electrostatic Energy

$$\boxed{\mathcal{E} = \int d^3x \ \frac{\varepsilon_0}{2} \vec{E}^2 = \frac{1}{2}\int d^3x \ \Phi(\vec{x})\rho(\vec{x})}$$

$$\boxed{= \frac{1}{8\pi\varepsilon_0}\int d^3x \, d^3x' \ \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

Capacitance

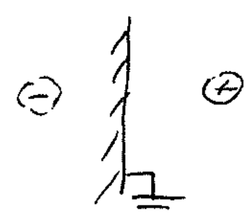
$$\boxed{Q_i = \sum_{j=1}^{n} C_{ij} V_j}$$



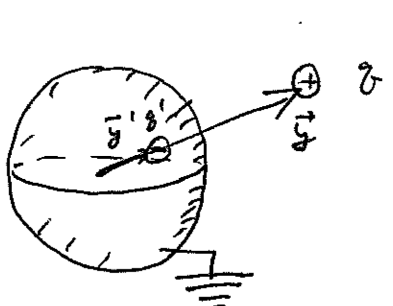
C_{ii} ~ capacitances

C_{ij}, $i \neq j$, coef's of induction

Method of Images:



Dirichlet Green function for a sphere

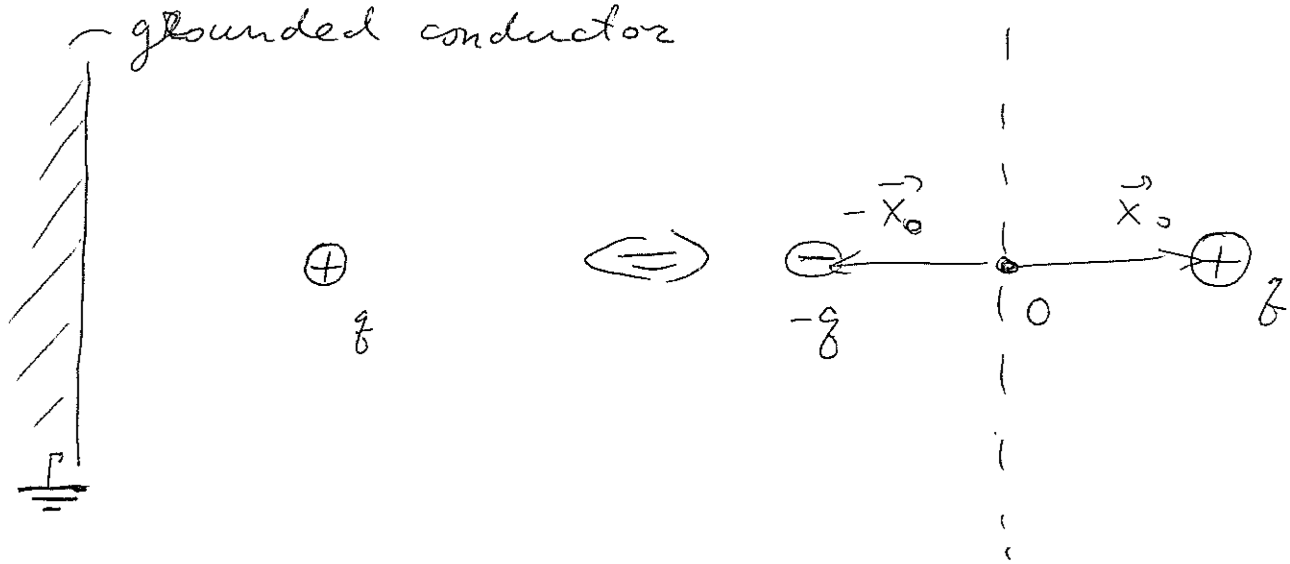


grounded conducting sphere

$$\Phi(\vec{x})\Big|_{|\vec{x}|=R} = 0$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}\right]$$

Method of Images.



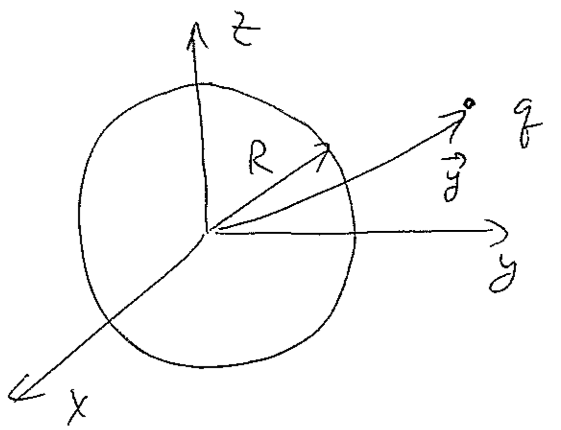
replace conductor by image charge(s):

$\phi = 0$ in conductor

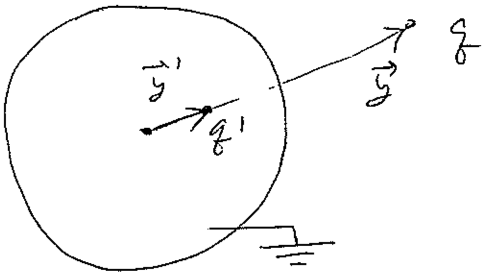
$$\phi_{\text{charge+image}} = \left(\frac{1}{|\vec{x} - \vec{x}_0|} - \frac{1}{|\vec{x} + \vec{x}_0|} \right) \frac{q}{4\pi\varepsilon_0}$$

$= 0$ if $\vec{x}$ is on (what used to be) boundary.

Dirichlet Green function for a Sphere.



First let's solve the problem of a point charge q & grounded conducting sphere at $\vec{y}$



The potential of charge q and image charge q' is

$$\Phi(\vec{x}) = \left(\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|} \right) \frac{1}{4\pi\varepsilon_0}$$

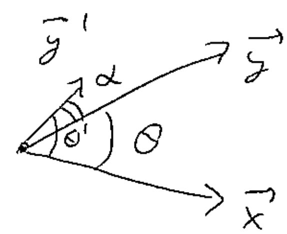
Let's demand $\Phi(|\vec{x}|=R) = 0$ (grounded conductor):

$$\left.\frac{q}{|\vec{x}-\vec{y}|}\right|_{|\vec{x}|=R} = -\left.\frac{q'}{|\vec{x}-\vec{y}'|}\right|_{|\vec{x}|=R}$$

$\Rightarrow$ sign q' = $-$ sign q

Square: (assume that $\vec{y}, \vec{y}'$ & $\vec{x}$ lie in same plane)

$$\left.\frac{q^2}{|\vec{x}-\vec{y}|^2}\right|_{|\vec{x}|=R} = \left.\frac{q'^2}{|\vec{x}-\vec{y}'|^2}\right|_{|\vec{x}|=R}$$



$$q^2\left(R^2 + y'^2 - 2Ry'\cos\theta'\right) = q'^2\left(R^2 + y^2 - 2Ry\cos\theta\right)$$

$\theta' = \theta + \alpha$ & identity should work for any θ

$\Rightarrow$ as $\cos\theta' = \cos(\theta+\alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$ (on l.h.s.)

$\Rightarrow$ no $\sin\theta$ on r.h.s. $\Rightarrow \sin\alpha = 0 \Rightarrow \alpha = 0, \pi; \cos\alpha = \pm 1$

$\Rightarrow \quad q^2(R^2+y'^2) = q'^2(R^2+y^2)$ and $\pm q^2 y' = q'^2 y$.

from $\cos\alpha$.

as $y, y' > 0$ (magnitudes of vectors) $\Rightarrow \cos\alpha = 1$

$\alpha = 0.$

$$\Rightarrow q^2 y' = q'^2 y \Rightarrow y' = \frac{q'^2 y}{q^2}$$

$$\Rightarrow q^2 R^2 + q^2 \frac{q'^4 y^2}{q^4} = q'^2 R^2 + q'^2 y^2$$

$$q'^4 \frac{y^2}{q^2} - q'^2 (R^2 + y^2) + q^2 R^2 = 0$$

$$q'^2 = \frac{1}{2 y^2/q^2} \left[+ R^2 + y^2 \pm \sqrt{(R^2+y^2)^2 - 4 y^2 R^2} \right] =$$

$$= \frac{q^2}{2y^2} \left[R^2 + y^2 \pm \underbrace{|R^2 - y^2|}_{= y^2 - R^2 \text{ as } y > R} \right]$$

$\Rightarrow$ ① $q'^2 = q^2$. and ② $q'^2 = q^2 \frac{R^2}{y^2}$.

① $\Rightarrow y' = y \Rightarrow$ put $-q$ on top of $q \Rightarrow$ get $\emptyset$:

however, $y' < R < y \Rightarrow$ can't have this case

② $\Rightarrow$ $\boxed{q' = -q \frac{R}{y}}$ $\boxed{y' = \frac{R^2}{y}}$ Mathematical transformation of inversion.

The potential is then

$$\boxed{\Phi(\vec{x}) = \left(\frac{q}{|\vec{x}-\vec{y}|} - q\,\frac{R}{y}\,\frac{1}{\left|\vec{x} - \frac{R^2}{y^2}\vec{y}\right|} \right)} \times \frac{1}{4\pi\varepsilon_0}$$

$\Rightarrow$ Dirichlet Green function is

$$\boxed{G_D(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} - \frac{R}{x'}\,\frac{1}{\left|\vec{x} - \frac{R^2}{x'^2}\vec{x}\,'\right|}}$$

<u>Exercise</u>: find the force between charge q & conducting sphere:

$$F = \frac{1}{4\pi\varepsilon_0}\,\frac{q\cdot q'}{|\vec{y}-\vec{y}\,'|^2} = -\frac{1}{4\pi\varepsilon_0}\,\frac{q^2 R/y}{\left[y - \frac{R^2}{y}\right]^2}.$$

The problem of finding $\Phi(\vec{x})$ for a charge q and any sphere is a Dirichlet problem:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x'\, G_D(\vec{x},\vec{x}\,')\,\rho(\vec{x}\,') - \frac{1}{4\pi}\oint_S da'\,\Phi(\vec{x}\,')\,\frac{\partial G_D(\vec{x},\vec{x}\,')}{\partial n'}.$$

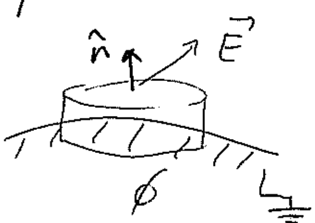
$\frac{\partial G_D}{\partial n'} = -\frac{\partial G_D}{\partial r'}$

Here $\Phi = 0$ on S for grounded conducting sphere.

<u>Exercise:</u> Find the surface charge density in a grounded conducting sphere in the presence of charge q:

we know $\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\vec{x}-\vec{y}|} - q\frac{R}{y}\frac{1}{|\vec{x} - \frac{R^2}{y^2}\vec{y}|} \right)$

sphere is conducting $\Rightarrow$ $\vec{E} = 0$ inside $\Rightarrow$



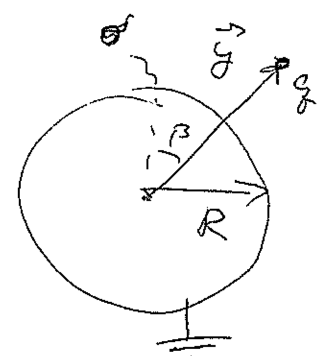
as discontinuity of E-field on the surface is $\Delta\vec{E}\cdot\hat{n} = \frac{\sigma}{\varepsilon_0}$, $\Delta E_t = 0$

$\Rightarrow$ $\frac{\sigma}{\varepsilon_0} = \vec{E}\cdot\hat{n}$, where $\vec{E}$ is the electric field outside the sphere $\Rightarrow$ as $\vec{E} = -\vec{\nabla}\phi \Rightarrow \frac{\sigma}{\varepsilon_0} = -\hat{n}\cdot\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r}$

$$\Rightarrow \sigma = -\varepsilon_0 \frac{\partial\phi}{\partial r}\Big|_{r=R} = -\frac{q}{4\pi}\left(\frac{\partial}{\partial r}\frac{1}{\sqrt{r^2+y^2-2ry\cos\beta}} - \frac{R}{y}\frac{\partial}{\partial r}\frac{1}{\sqrt{r^2+\frac{R^4}{y^2} - 2r\frac{R^2}{y}\cos\beta}} \right)\Bigg|_{r=R} = -\frac{q}{4\pi}\frac{1}{R}\frac{y^2-R^2}{\left(y^2+R^2-2Ry\cos\beta\right)^{3/2}}.$$

$\Rightarrow$ $\sigma < 0$ everywhere as $y > R$

$\Rightarrow$ total charge on the sphere $Q = \oint_S da\,\sigma < 0$ $\quad = -q\,R/y = \underline{\underline{q'}}$



Is there any contradiction?

If not, why?

$$\lambda = \int_{-1}^{1} d\cos\theta \cdot 2\pi \cdot R^2 \sigma = -\frac{q}{2} R \int_{-1}^{1} d\cos\theta \, \frac{y^2 - R^2}{\left[y^2 + R^2 - 2Ry\cos\theta\right]^{3/2}}$$

$$= -\frac{q}{2} \cdot R \cdot (y^2 - R^2) \cdot \left. \frac{1}{\left[y^2 + R^2 - 2Ry\cos\theta\right]^{1/2}} \right|_{-1}^{1} \frac{1}{Ry} =$$

$$= -\frac{q}{2} \cancel{R} \frac{1}{y} (y^2 - R^2) \left(\frac{1}{y - R} - \frac{1}{y + R} \right) =$$

$$= -\frac{q}{\cancel{2}} \frac{1}{y} \cancel{(y^2 - R^2)} \frac{\cancel{2}R}{\cancel{y^2 - R^2}} = -\frac{qR}{y} = q'$$

$\Rightarrow$ induced charge = image charge

(as expected from Gauss's Law)

Suppose the sphere has a total charge Q $\Rightarrow$

split it into q' & $Q-q'$ [alternatively: start with a grounded sphere and charge q: disconnect the ground; you now have q' on the sphere and $\phi=0$ then add $Q-q'$.

$\Rightarrow$ In conductors all charge sits on the surface

bring Q from infinity: first bring q' then $Q-q'$.

$\Rightarrow$ q' along with q creates $\phi=0$ on surface

$\Rightarrow$ $Q-q'$ is uniformly distributed on the surface, giving extra

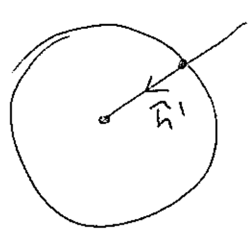
$$\Delta\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{Q-q'}{|\vec{x}|} = \frac{1}{4\pi\varepsilon_0}\frac{Q+q\,R/y}{|\vec{x}|}$$

in addition to the potential found above, such that total potential is

$$\phi(\vec{x}) = \left[\frac{q}{|\vec{x}-\vec{y}|} - q\frac{R}{y}\frac{1}{\left|\vec{x}-\frac{R^2}{y^2}\vec{y}\right|} + \frac{Q+q\,R/y}{|\vec{x}|}\right]\frac{1}{4\pi\varepsilon_0}$$

For general boundary condition $\phi(R,\theta,\varphi)$ on the surface of the sphere, to find the potential

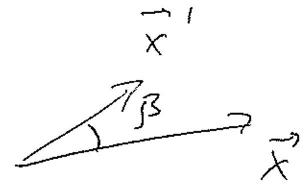
we need $\left.\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\right|_{|\vec{x}'|=R} = -\left.\frac{\partial G_D}{\partial r'}\right|_{r'=R}$ $\Rightarrow$



Write

$$G_D(\vec{x},\vec{x}') = \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\beta}} -$$

$$- \frac{R}{x'} \frac{1}{\sqrt{x^2+\frac{R^4}{x'^2}-2x\frac{R^2}{x'}\cos\beta}}$$



$$\Rightarrow -\left.\frac{\partial G_D}{\partial x'}\right|_{x'=R} = \left\{ \frac{x'-x\cos\beta}{\left(\sqrt{x^2+x'^2-2xx'\cos\beta}\right)^3} - R \cdot \right.$$

$$\left. \cdot \left. \frac{x'x^2-xR^2\cos\beta}{\left[x^2x'^2+R^4-2xx'R^2\cos\beta\right]^{3/2}} \right\} \right|_{x'=R} =$$

$$= \frac{R - x^2/R}{\left(x^2+R^2-2xR\cos\beta\right)^{3/2}}$$

outside charges

$$\Rightarrow \boxed{\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(\vec{x}')\,G_D(\vec{x},\vec{x}') - - \frac{1}{4\pi}\int d\Omega'\,\frac{R(R^2-x^2)}{(x^2+R^2-2xR\cos\beta')^{3/2}}\,\phi(R,\theta',\varphi')}$$

where $d\Omega = d\cos\theta\, d\varphi$