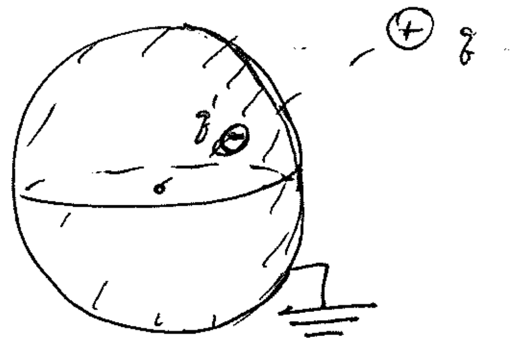
Last time | We finished constructing Dirichlet Green function for a sphere (outside a sphere)

$$\boxed{G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{R}{x'} \frac{1}{\left|\vec{x} - \frac{R^2}{x'^2}\vec{x}'\right|}}$$

using the method of images:



grounded conducting sphere

Employing the solution of this problem we found the force on charge q:

(attractive)

$$F = \frac{-1}{4\pi\varepsilon_0} \frac{q^2 R/y}{\left[y - \frac{R^2}{y}\right]^2}$$

and the induced charge density on the sphere

$$\sigma = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=R} = \frac{-q}{4\pi R} \frac{y^2 - R^2}{\left[y^2 + R^2 - 2Ry\cos\theta\right]^{3/2}}$$

where θ is the angle between $\vec{y}$ & $\vec{x}$.

$\Rightarrow$ Note: the sphere is at $\Phi = 0 \Rightarrow$ if we now disconnect the ground, nothing would change

$\Rightarrow$ can bring in a charge $Q - q'$ and place it on the sphere $\Rightarrow$ would distribute itself uniformly on the surface

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} - q\frac{R}{y}\frac{1}{|\vec{x}-\frac{R^2}{y^2}\vec{y}|} + \frac{Q+q\,R/y}{|\vec{x}|}\right]$$

Suppose the sphere has a total charge Q $\Rightarrow$

split it into q' & $Q-q'$

[alternatively: start with a grounded sphere and charge q: disconnect the ground; you now have q' on the sphere and $\Phi=0$ then add $Q-q'$.]

$\Rightarrow$ In conductors all charge sits on the surface

bring Q from infinity: first bring q' then $Q-q'$.

$\Rightarrow$ q' along with q creates $\phi = 0$ on surface

$\Rightarrow$ $Q-q'$ is uniformly distributed on the surface, giving extra

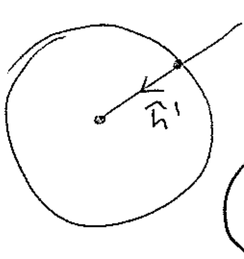
$$\Delta\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{Q-q'}{|\vec{x}|} = \frac{1}{4\pi\varepsilon_0}\frac{Q+q\,R/y}{|\vec{x}|}$$

in addition to the potential found above, such that total potential is

$$\Phi(\vec{x}) = \left[\frac{q}{|\vec{x}-\vec{y}|} - q\frac{R}{y}\frac{1}{\left|\vec{x}-\frac{R^2}{y^2}\vec{y}\right|} + \frac{Q+q\,R/y}{|\vec{x}|}\right]\frac{1}{4\pi\varepsilon_0}$$

For general boundary condition $\Phi(R,\theta,\varphi)$ on the surface of the sphere, to find the potential we need

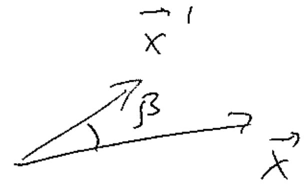
$$\left.\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\right|_{|\vec{x}'|=R} = -\left.\frac{\partial G_D}{\partial r'}\right|_{r'=R} \Rightarrow$$



General ~~formula~~: (Dirichlet Green function)

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x'\, G_D(\vec{x},\vec{x}')\,\rho(\vec{x}') - \frac{1}{4\pi}\oint_S da'\,\Phi(\vec{x}')\,\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}$$

Write
$$G_D(\vec{x},\vec{x}') = \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\beta}} - \frac{R}{x'}\,\frac{1}{\sqrt{x^2+\frac{R^4}{x'^2}-2x\frac{R^2}{x'}\cos\beta}}$$



$$\Rightarrow -\frac{\partial G_D}{\partial x'}\bigg|_{x'=R} = \left\{ \frac{x'-x\cos\beta}{\left(\sqrt{x^2+x'^2-2xx'\cos\beta}\right)^3} - R\cdot \frac{x'x^2 - xR^2\cos\beta}{\left[x^2x'^2+R^4-2xx'R^2\cos\beta\right]^{3/2}} \right\}\Bigg|_{x'=R} =$$

$$= \frac{R - x^2/R}{\left(x^2+R^2-2xR\cos\beta\right)^{3/2}}$$

outside charges

$$\Rightarrow \boxed{\begin{aligned} \Phi(\vec{x}) &= \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(\vec{x}')\,G_D(\vec{x},\vec{x}') - \\ &- \frac{1}{4\pi}\int d\Omega'\,\frac{R(R^2-x^2)}{(x^2+R^2-2xR\cos\beta')^{3/2}}\,\Phi(R,\theta',\varphi') \end{aligned}}$$

where $d\Omega = d\cos\theta\, d\varphi$

and β = angle between $\vec{x}$ and $\vec{x}'$

If $\Phi(R, \theta', \varphi') = V$ a constant & $\rho = 0$

$$\Rightarrow \quad \Phi(\vec{x}) = \frac{1}{4\pi} \int \underbrace{d\Omega}_{= \, d\varphi \, d\cos\beta} \, V \frac{R\,(x^2 - R^2)}{(x^2 + R^2 - 2xR\cos\beta)^{3/2}} =$$

$$= \frac{1}{4\pi} \cdot 2\pi \cdot \int_{-1}^{1} d\cos\beta \; \frac{R\,(x^2 - R^2)}{(x^2 + R^2 - 2xR\cos\beta)^{3/2}} \, V =$$

$$= \frac{1}{2} \, \frac{R\,(x^2 - R^2)}{x\cancel{R}} \, V \, \frac{1}{(x^2 + R^2 - 2xR\cos\beta)^{1/2}} \Bigg|_{-1}^{1} =$$

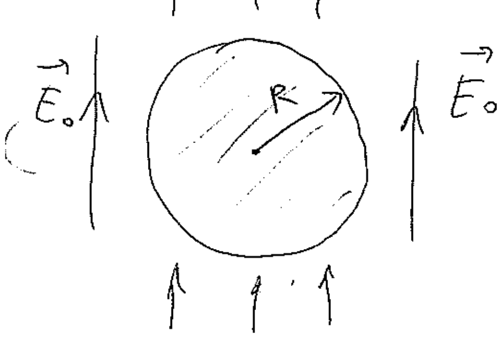
$$= \frac{1}{2} \, \frac{x^2 - R^2}{x} \, V \left(\frac{1}{|x - R|} - \frac{1}{|x + R|} \right) = \text{as } x > R =$$

$$= V \frac{R}{x} \quad \Rightarrow \quad \Phi(\vec{x}) = V \frac{R}{x}$$

as one'd expect from Gauss's law.

(skip)

Consider a conducting sphere in a uniform electric field $\vec{E}_0$: let's guess the answer for ϕ!



$$\phi(\vec{r}) = \underbrace{-\vec{E}_0 \cdot \vec{r}}_{\text{potential due to field } \vec{E}_0} + \underbrace{\phi_{sphere}(\vec{r})}_{\text{potential due to the sphere.}}$$

Laplace equation $\nabla^2 \phi = 0$ is valid everywhere outside of the sphere $\Rightarrow$

$$\Rightarrow \quad \nabla^2 \phi_{sphere}(\vec{r}) = 0.$$

$\phi_{sphere}(\vec{r})$ depends on r and $\vec{E}_0$ & satisfies $\nabla^2 \phi_{sphere} = 0$ (& vanishes at ∞) $\Rightarrow$ $\phi_{sphere}(\vec{r}) \propto \vec{E}_0 \cdot \vec{\nabla} \frac{1}{r}$

is a natural guess $\Rightarrow$ as $\vec{\nabla} \frac{1}{r} = -\frac{\vec{r}}{r^3}$ $\Rightarrow$

$$\phi(\vec{r}) = -\vec{E}_0 \cdot \vec{r} + \underset{\uparrow \text{ constant}}{C} \cdot \vec{E}_0 \cdot \frac{\vec{r}}{r^3}$$

Require that $\phi(\vec{r})\Big|_{|\vec{r}|=R} = 0 \Rightarrow -1 + \frac{C}{R^3} = 0$

$$\Rightarrow C = R^3 \Rightarrow \boxed{\phi(\vec{r}) = -\vec{E}_0 \cdot \vec{r}\left(1 - \frac{R^3}{r^3}\right)}$$

(cf. Jackson's Eq. (2.14)).

Surface charge density

$$\sigma = -\varepsilon_0 \frac{\partial \phi}{\partial r}\Big|_{r=a} = \varepsilon_0 E_0 \cos\theta + 2\varepsilon_0 E_0 \cos\theta = 3\varepsilon_0 E_0 \cos\theta$$

$$(\Rightarrow Q = \oint_S da\, \sigma = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \; 3\varepsilon_0 E_0 \cos\theta = 0$$

$\Rightarrow$ sphere could be insulated or grounded...