We have: ⇒ derived Poisson/Laplace equations

⇒ formulated b.c. problems: Dirichlet & Neumann

⇒ showed that solutions can be expressed in terms of Green functions

⇒ studied one way to find Green functions ~ ~ method of images.

⇒ is there any other way to solve Poisson/Laplace equation? to find Green functions?

Separation of Variables

Orthogonal Functions

Def.

Orthonormal set of functions on $[a,b]$ is defined by

$$\boxed{\int_a^b dx\, u_n^*(x)\, u_m(x) = \delta_{mn}}$$

$\{u_n(x)\}$, $n>0$, $n \in$ integers

where $\delta_{mn} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$ Kronecker delta.

Our goal is to approximate any function

$$f(x) \Leftrightarrow \sum_{n=1}^{N} a_n u_n(x) \quad \text{(partial sums)}$$

such that for $N \to \infty$

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x).$$

the coefficients can be determined by multiplying the equality by $u_m^*(x)$ & integrating:

$$\int_a^b dx\, u_m^*(x) f(x) = \sum_{n=1}^{\infty} a_n \underbrace{\int_a^b dx\, u_m^*(x)\, u_n(x)}_{\delta_{mn}} = a_m$$

$$\Rightarrow \quad a_n = \int_a^b dx\, f(x)\, u_n^*(x).$$

Def. The set $\{u_n(x)\}$ is <u>complete</u> on $[a,b]$ if any ("good") function $f(x)$ can be expanded on $[a,b]$ in $\sum_{n=1}^{\infty} a_n u_n(x)$ (or, more precisely, being successfully approximated by partial sums $\sum_{n=1}^{N} a_n u_n(x)$).

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x) = \sum_{n=1}^{\infty} \int_a^b dx'\, f(x')\, u_n^*(x')\, u_n(x)$$

$$= \int_a^b dx'\, f(x') \underbrace{\sum_{n=1}^{\infty} u_n^*(x')\, u_n(x)}_{\text{acts like } \delta\text{-ftn},\ \propto\, \delta(x-x')}$$

$$\Rightarrow \quad \boxed{\sum_{n=1}^{\infty} u_n^*(x')\, u_n(x) = \delta(x-x').}$$

<u>completeness relation</u>.