Last time | Finished method of images.

Separation of Variables

Orthogonal Functions (cont'd)

Def. **Orthonormal** set of functions: $\{u_n(x)\}$, $n>0$, n = integer

$$\boxed{\int_a^b dx\, u_n^*(x)\, u_m(x) = \delta_{nm}}$$ orthogonality

Want to expand any function over this set:

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x)$$

where $$a_n = \int_a^b dx\, f(x)\, u_n^*(x)$$

For such expansion to exist for $\forall f(x)$ need completeness:

Def. Set is complete if

$$\boxed{\sum_{n=1}^{\infty} u_n^*(x')\, u_n(x) = \delta(x-x')}$$ completeness

$$\hat{L}_x \Phi(\vec{x}) = -\rho(\vec{x})/\epsilon_0 \quad \text{and} \quad \hat{L}_x u_n(\vec{x}) = \lambda_n u_n(\vec{x})$$

← linear operator

↑ eigenfunctions $\{u_n(\vec{x})\}$ complete orthonormal set

← eigenvalues

$$\int d^3x\, u_n(\vec{x})\, u_m^*(\vec{x}) = \delta_{nm}$$

⇒ look for Green ftn:

$$\hat{L}_x G(\vec{x},\vec{x}') = -4\pi\, \delta^3(\vec{x}-\vec{x}')$$

$$G(\vec{x},\vec{x}') = \sum_n a_n(\vec{x}')\, u_n(\vec{x})$$

$$\Rightarrow \quad \hat{L}_x G(\vec{x},\vec{x}') = \sum_n a_n(\vec{x}')\, \lambda_n u_n(\vec{x}) = -4\pi\, \delta^3(\vec{x}-\vec{x}')$$

multiply by $u_m^*(\vec{x})$ & integrate over $\vec{x}$:

$$a_m(\vec{x}')\lambda_m = -4\pi\, u_m^*(\vec{x}') \Rightarrow a_m(\vec{x}') = -4\pi \frac{u_m^*(\vec{x}')}{\lambda_m}$$

$$\Rightarrow \boxed{G(\vec{x},\vec{x}') = -4\pi \sum_m \frac{u_m(\vec{x})\, u_m^*(\vec{x}')}{\lambda_m}}$$

Note that $\boxed{\delta^3(\vec{x}-\vec{x}') = \sum_n u_n(\vec{x})\, u_n^*(\vec{x}')}$ (completeness)

⇒ Can find the Green function as an expansion over a complete & orthogonal set of functions $\{u_n(\vec{x})\}$.

Famous example: for $x \in \left(-\frac{a}{2}, \frac{a}{2}\right)$

sines & cosines

$$\{u_m(x)\} \Longleftrightarrow \left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi m x}{a}\right), \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi m x}{a}\right) \right\}$$

$m \geq 0$

Fourier expansion:

$$f(x) = \frac{1}{2} A_0 + \sum_{m=1}^{\infty} \left[A_m \cos\left(\frac{2\pi m x}{a}\right) + B_m \sin\left(\frac{2\pi m x}{a}\right) \right]$$

where
$$\begin{pmatrix} A_m \\ B_m \end{pmatrix} = \frac{2}{a} \int_{-a/2}^{a/2} dx\, f(x) \begin{pmatrix} \cos\left(\frac{2\pi m x}{a}\right) \\ \sin\left(\frac{2\pi m x}{a}\right) \end{pmatrix}$$

Check that sines and cosines form complete orthonormal set:

$$\int_{-a/2}^{a/2} dx\, \frac{2}{a} \sin\left(\frac{2\pi m x}{a}\right) \sin\left(\frac{2\pi n x}{a}\right) =$$

$$= \frac{2}{a} \int_{-a/2}^{a/2} dx\, \frac{1}{2} \left[-\cos\left(\frac{2\pi x}{a}(m+n)\right) + \cos\left(\frac{2\pi x}{a}(m-n)\right) \right] =$$

$$= \begin{cases} 1, & m = n \neq 0 \\ 0, & m \neq n,\ m, n > 0 \end{cases}$$

ibid for cosines, etc.

To check completeness let's use complex exponents instead: $u_m(x) = \frac{1}{\sqrt{a}} e^{i \frac{2\pi m x}{a}}$.

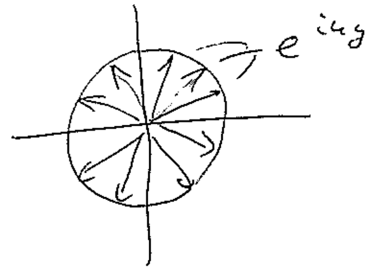
now $m = (-\infty \ldots +\infty)$, $m = 0, \pm 1, \pm 2, \ldots$

Need to see that $\sum_{n=-\infty}^{\infty} u_n^*(x') u_n(x) = \delta(x-x')$

$$\sum_{n=-\infty}^{\infty} u_n^*(x')u_n(x) = \frac{1}{a}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{a}(x-x')}$$

Define $y = \frac{2\pi}{a}(x-x') \Rightarrow \frac{1}{a}\sum_{n=-\infty}^{+\infty} e^{in\cdot y} = \begin{cases} 0, & y\neq 0 \\ \infty, & y=0 \end{cases}$

for $y\neq 0$ go to complex plane:



sum of vectors $\simeq 0$.

$\Rightarrow$ condition (i) is satisfied

to check (ii) we integrate

$$\int_{-a/2}^{a/2} dx \cdot \frac{1}{a}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{a}\cdot x} = \frac{1}{a}\sum_{n=-\infty}^{\infty}\frac{a}{i2\pi n}\left(e^{i\pi n}-e^{-i\pi n}\right) =$$

$$= \sum_{n=-\infty}^{\infty}\frac{1}{2\pi n i}\, 2i\sin(\pi n) = \sum_{n=-\infty}^{\infty}\frac{\overbrace{\sin(\pi n)}^{0 \text{ for } n\neq 0}}{\pi n} = 1$$

(only $n=0$ contributes)

$\Rightarrow$ the set is complete!

$$f(x) = \sum_m A_m u_m(x) = \frac{1}{\sqrt{a}}\sum_m A_m e^{i\frac{2\pi m x}{a}}$$

Def. <u>Fourier integral:</u> replace $\sum_m \to \int_{-\infty}^{\infty} dm$

($\frac{2\pi m}{a} \to k$, $A_m \to \sqrt{\frac{2\pi}{a}}\, A(k)$ (Jackson's convention)

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik\cdot x}$$ Fourier integral / transform

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ik\cdot x}$$ inverse Fourier transform.

orthogonality condition becomes:

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = \delta(k-k')}$$

while the completeness relation is

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x-x')}$$

Let's prove it:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{iq\cdot x} = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{iq\cdot x - \varepsilon^2 x^2} =$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-\varepsilon^2 \left(x - \frac{iq}{2\varepsilon^2}\right)^2}}_{\sqrt{\pi/\varepsilon^2}} e^{-\frac{q^2}{4\varepsilon^2}} =$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\sqrt{\pi}\,\varepsilon} e^{-q^2/4\varepsilon^2} = \frac{1}{2}\, \delta\!\left(\frac{q}{2}\right) = \delta(q)$$

as desired!

representation of δ-ftn studied before.