Last time | **Orthogonal Functions** (cont'd)

$\{u_n(\vec{x})\}$ ~ orthonormal complete set

Showed that if the set is made out of eigenfunction of operator $\hat{L}_x$ $\Rightarrow$ the Green function for this operator can be written as

$$\boxed{G(\vec{x},\vec{x}\,') = -4\pi \sum_n \frac{u_n(\vec{x})\, u_n^*(\vec{x}\,')}{\lambda_n}}$$

where λ_n are the eigenvalues : $\hat{L}_x u_n(\vec{x}) = \lambda_n u_n(\vec{x})$

Example | $u_n(x) = \frac{1}{\sqrt{a}} e^{i\frac{2\pi n x}{a}}$ on $x \in [-\frac{a}{2}, \frac{a}{2}]$

$\Rightarrow$ showed that this is an orthonormal and complete set:

$$\sum_{n=-\infty}^{\infty} u_n^*(x')\, u_n(x) = \delta(x-x')$$

That is, $\frac{1}{a} \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{a}(x-x')} = \delta(x-x')$ on $[-\frac{a}{2}, \frac{a}{2}]$

Considered functions $\{e^{i\vec{k}\cdot\vec{x}}\}$ normalized as

$$\int d^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

completeness: $$\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')} = \delta^3(\vec{x}-\vec{x}\,')$$

$$\Rightarrow \boxed{f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} A(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}}$$ ~ Fourier integral

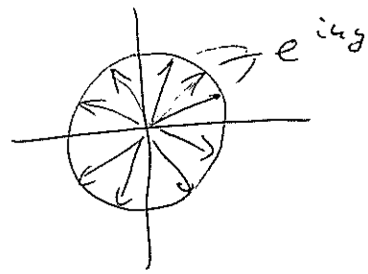
coefficients: $$\boxed{A(\vec{k}) = \int d^3x\, f(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}}$$

Need to see that $\sum_{n=-\infty}^{\infty} u_n^*(x')u_n(x) = \delta(x-x')$

$$\sum_{n=-\infty}^{\infty} u_n^*(x')u_n(x) = \frac{1}{a}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{a}(x-x')}$$

Define $y = \frac{2\pi}{a}(x-x') \Rightarrow \frac{1}{a}\sum_{n=-\infty}^{+\infty} e^{in\cdot y} = \begin{cases} 0, & y\neq 0 \\ \infty, & y=0 \end{cases}$

for $y \neq 0$ go to complex plane:



sum of vectors $\simeq 0$.

$\Rightarrow$ condition (i) is satisfied

to check (ii) we integrate

$$\int_{-a/2}^{a/2} dx \cdot \frac{1}{a}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{a}(x-x')} = \frac{1}{a}\sum_{n=-\infty}^{\infty} \frac{a e^{-i\frac{2\pi n}{a}x'}}{i2\pi n}\left(e^{i\pi n} - e^{-i\pi n}\right) =$$

$$= \sum_{n=-\infty}^{\infty} \frac{e^{-\frac{2\pi n}{a}ix'}}{2\pi n i} 2i\sin(\pi n) = \sum_{n=-\infty}^{\infty} e^{-i\frac{2\pi n}{a}x'} \frac{\overbrace{\sin(\pi n)}^{0 \text{ for } n\neq 0}}{\pi n} = 1$$

(only $n=0$ contributes)

$\Rightarrow$ the set is complete!

$$f(x) = \sum_m A_m u_m(x) = \frac{1}{\sqrt{a}}\sum_m A_m e^{i\frac{2\pi m x}{a}}$$

Def. **Fourier integral:** replace $\sum_m \to \int_{-\infty}^{\infty} dm = \frac{a}{2\pi}\int_{-\infty}^{\infty} dk$

$\frac{2\pi m}{a} \to k$, $A_m \to \sqrt{\frac{2\pi}{a}}\, A(k)$ (Jackson's convention)

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik\cdot x}$$ Fourier integral / transform

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ik\cdot x}$$ inverse Fourier transform.

orthogonality condition becomes: (the functions now are $u_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$)

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = \delta(k-k')}$$

while the completeness relation is

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x-x')}$$

Let's prove it:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{iq\cdot x} = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{iq\cdot x - \varepsilon^2 x^2} =$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-\varepsilon^2\left(x - \frac{iq}{2\varepsilon^2}\right)^2}}_{\sqrt{\pi/\varepsilon^2}} e^{-\frac{q^2}{4\varepsilon^2}} =$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\sqrt{\pi}\,\varepsilon} e^{-q^2/4\varepsilon^2} = \frac{1}{2}\,\delta\!\left(\frac{q}{2}\right) = \delta(q)$$

as desired!

representation of δ-ftn studied before

$$\frac{1}{2\pi}\left[\int_0^{\infty} dx\, e^{i(q+i\varepsilon)x} + \int_{-\infty}^{0} dx\, e^{i(q-i\varepsilon)x}\right] = \frac{1}{2\pi}\left[\frac{i}{q+i\varepsilon} - \frac{i}{q-i\varepsilon}\right] = \frac{i}{2\pi}(-2\pi i)\delta(q) = \delta(q).$$

For many dimensions:

$$\delta(\vec{x}-\vec{x}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z') = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{ik_x\cdot(x-x')}$$

$$\cdot \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} e^{ik_y\cdot(y-y')} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{ik_z\cdot(z-z')} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$\Rightarrow$ to solve $\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$ write

$$G(\vec{x},\vec{x}') = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \cdot \tilde{G}(\vec{k})$$

$$\Rightarrow \nabla^2 G(\vec{x},\vec{x}') = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} (-\vec{k}^2)\tilde{G}(\vec{k})$$

$$= -4\pi\,\delta(\vec{x}-\vec{x}') = -4\pi \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$\Rightarrow -\vec{k}^2 G(\vec{k}) = -\frac{4\pi}{(2\pi)^{3/2}} \Rightarrow \boxed{G(\vec{k}) = \frac{4\pi}{(2\pi)^{3/2}} \frac{1}{\vec{k}^2}}$$

Such that

$$G(\vec{x},\vec{x}') = \int \frac{d^3k}{2\pi^2} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \frac{1}{\vec{k}^2}$$

cf. $G(\vec{x},\vec{x}') = -4\pi \cdot \sum_n \frac{u_n(\vec{x})\,u_n^*(\vec{x}')}{\lambda_n}$.

Here $\lambda_n = -\vec{k}^2$

$\Rightarrow$ going to Fourier space is a powerfull method for solving differential equations (an example of eigenfunction expansion)

Check: $\frac{1}{2\pi^2}\int \frac{d^3k}{\vec{k}^2}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} =$ | go to spherical coordinates, with $\vec{x}-\vec{x}'$ pointing in z-direction

$$= \frac{1}{2\pi^2}\int_0^\infty dk\cdot \cancel{k^2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_{-1}^{1} d\cos\theta\, \frac{1}{\cancel{k^2}}\, e^{ik|\vec{x}-\vec{x}'|\cos\theta} =$$

$$= \frac{1}{\pi}\int_0^\infty dk\cdot \frac{1}{ik|\vec{x}-\vec{x}'|}\underbrace{\left[e^{ik\cdot|\vec{x}-\vec{x}'|} - e^{-ik\cdot|\vec{x}-\vec{x}'|}\right]}_{2i\sin(k|\vec{x}-\vec{x}'|)} =$$

$$= \frac{2}{\pi}\,\frac{1}{|\vec{x}-\vec{x}'|}\underbrace{\int_0^\infty \frac{dk}{k}\cdot \sin(k|\vec{x}-\vec{x}'|)}_{=\frac{\pi}{2}\text{ (see Solution of h/w 1)}} = \frac{1}{|\vec{x}-\vec{x}'|} \text{ as desired.}$$

Separation of Variables (cont'd)

a powerfull new tool for solving Poisson/Laplace equations. Depending on geometry we'll consider three main cases: separation of variables in rectangular, spherical & cylindrical coordinates.

Laplace / Poisson Equation in rectangular coordinates.

good for problems involving fields / potential in a box. (to get "outside the box" ~ see spherical & cylindrical cases).

Start with Laplace equation in rectangular coordinates: $\nabla^2 \Phi = 0$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Look for solution in the form $\Phi(x,y,z) = X(x)\,Y(y)\,Z(z)$

Plug it in: $X'' Y Z + X Y'' Z + X Y Z'' = 0$

$$\frac{X''}{X}(x) + \frac{Y''}{Y}(y) + \frac{Z''}{Z}(z) = 0$$

Should work for any x, y, z $\Rightarrow$

$$\begin{cases} \dfrac{X''}{X} = -\alpha^2 & \Rightarrow X(x) = C_1 e^{i\alpha x} + C_2 e^{-i\alpha x} \\ \dfrac{Y''}{Y} = -\beta^2 & \Rightarrow Y(y) = \tilde{C}_1 e^{i\beta y} + \tilde{C}_2 e^{-i\beta y} \\ \dfrac{Z''}{Z} = \alpha^2 + \beta^2 \equiv \gamma^2 & \Rightarrow Z(z) = \tilde{\tilde{C}}_1 e^{\gamma z} + \tilde{\tilde{C}}_2 e^{-\gamma z} \end{cases}$$

general solution of Laplace eqn.

(13) Eigenfunctions of ∇^2 operator in rectangular

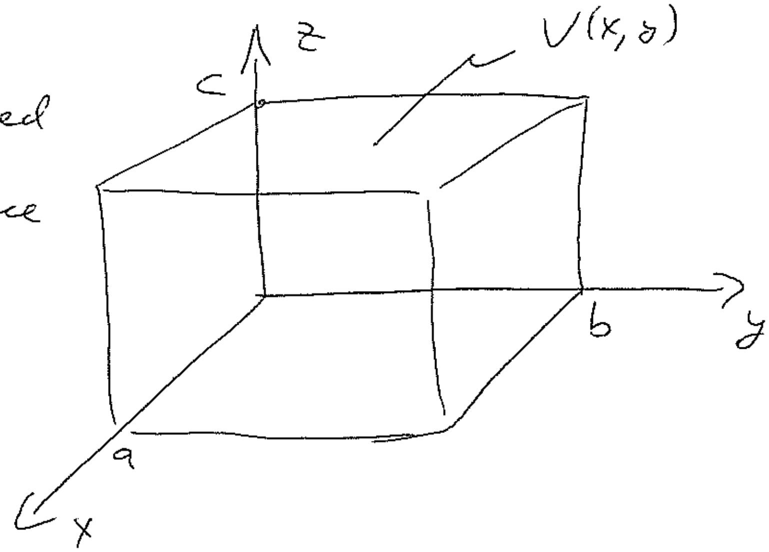
coordinates are exponents $e^{I a x_i}$, where a can be real or imaginary, and $x_1 = x$, $x_2 = y$, $x_3 = z$.

$\Rightarrow$ General strategy: use separation of variables to find eigenfunction of ∇^2 operator in various coordinates.

Let's consider an example: a box:

all surfaces grounded exept the top surface sitting at potential $V(x, y)$.

(note: $V(0,y) = V(a,y) = 0$
$V(x,0) = V(x,b) = 0$)



$X(0) = 0 \Rightarrow X(x) \propto \sin(\alpha x)$

$Y(0) = 0 \Rightarrow Y(y) \propto \sin(\beta y)$

$Z(0) = 0 \Rightarrow Z(z) \propto \sinh(\gamma z)$

$X(a) = 0 \Rightarrow \sin(\alpha a) = 0 \Rightarrow \alpha_n = \frac{\pi n}{a}$

$Y(b) = 0 \Rightarrow \beta_n = \frac{\pi m}{b}$.

$$\Rightarrow \gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2} = \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2} = \gamma_{nm}$$

$$\Rightarrow \Phi_{nm}(x,y,z) \propto \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

$$\Rightarrow \Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z).$$

Finally, $\Phi(x, y, z=c) = V(x,y) \Rightarrow$

$$\Rightarrow V(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} c)$$

It's a double Fourier Series $\Rightarrow$ can invert obtaining

$$A_{nm} = \frac{4}{ab\sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy\, V(x,y)\sin(\alpha_n x)\sin(\beta_m y).$$

problem solved!

To find a general solution for Laplace/Poisson equations with Dirichlet boundary conditions we need to find Green function $G_D(\vec{x}, \vec{x}')$. The construction is similar to using the Fourier transform & we need to solve $\nabla^2 G_D(\vec{x}, \vec{x}') = -4\pi\,\delta(\vec{x} - \vec{x}')$ and find G_D that vanishes on the boundary.

Method I : expansion in sines.

Need to solve $\nabla'^2 G_D(\vec{x},\vec{x}\,') = -4\pi\,\delta^3(\vec{x}-\vec{x}\,')$ with

$G_D(\vec{x},\vec{x}\,') = 0$ for $\vec{x}\,'$ on the boundary of the box.

Look for G_D in the following form:

$$G_D(\vec{x},\vec{x}\,') = \sum_{l,m,n=1}^{\infty} G_{lmn}(\vec{x}) \sin\left(\frac{\pi l x'}{a}\right) \sin\left(\frac{\pi m y'}{b}\right) \sin\left(\frac{\pi n z'}{c}\right)$$

$\Rightarrow$ as $G_D(\vec{x},\vec{x}\,') = G_D(\vec{x}\,',\vec{x})$ ~ symmetric $\Rightarrow$

$$G_D(\vec{x},\vec{x}\,') = \sum_{l,m,n=1}^{\infty} \underbrace{\frac{8}{abc} G_{lmn}}_{\text{new coeff.}} \sin\left(\frac{\pi l x}{a}\right) \sin\left(\frac{\pi l x'}{a}\right) \sin\left(\frac{\pi m y}{b}\right) \cdot$$

$$\cdot \sin\left(\frac{\pi m y'}{b}\right) \sin\left(\frac{\pi n z}{c}\right) \sin\left(\frac{\pi n z'}{c}\right).$$

To solve $\nabla'^2 G_D(\vec{x},\vec{x}\,') = -4\pi\,\delta^3(\vec{x}-\vec{x}\,')$ need to expand the δ-functions in sines too.

Above we showed that

$$\{u_n(x)\} = \left\{\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi n x}{a}\right), \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi m x}{a}\right)\right\}$$

is a complete set on $x \in \left(-\frac{a}{2}, \frac{a}{2}\right)$. $\Rightarrow$ it is also a complete set on $x \in (0,a)$. However, on $x \in (0,a)$ we can use a different complete set of functions:

$$\left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right) \right\}, \quad n = 1, 2, 3, \ldots$$

Clearly the functions are orthogonal:

$$\int_0^a dx \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right) \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n' x}{a}\right) = \delta_{nn'}.$$

To prove completeness write

$$\frac{2}{a}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right) = -\frac{1}{a}\sum_{n=1}^{\infty}\left[e^{i\frac{\pi n}{a}(x+x')} + \right.$$

$$\left. + e^{-i\frac{\pi n}{a}(x+x')} - e^{i\frac{\pi n}{a}(x-x')} - e^{-i\frac{\pi n}{a}(x-x')}\right] =$$

$$= -\frac{1}{a}\sum_{n=-\infty}^{\infty}\left[e^{i\frac{\pi n}{a}(x+x')} - e^{i\frac{\pi n}{a}(x-x')}\right] = \Bigg|$$ can prove similar to above

$$= \begin{cases} 0, & x+x' \neq 0,\ x-x' \neq 0 \\ \mp\infty, & x+x'=0 \text{ or } x-x'=0 \end{cases}$$

& note that $x>0, x'>0$
$\Rightarrow x+x'>0$,
never $=0$.

$$\Rightarrow \frac{2}{a}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right) \propto \delta(x-x') \text{ for}$$

$$0 < x, x' < a.$$

To fix the coefficient we integrate:

$$\frac{2}{a}\sum_{n=1}^{\infty} \int_0^a dx \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right) = \sum_{n=1}^{\infty} \frac{-2}{a}\frac{a}{\pi n}\left[\cos(\pi n) - 1\right]$$

$$\cdot \sin\left(\frac{\pi n x'}{a}\right) = -2\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n} \sin\left(\frac{\pi n x'}{a}\right) = \Bigg|$$ only odd n survive $n = 2m+1$

$$= 4 \sum_{m=0}^{\infty} \frac{1}{\pi(2m+1)} \sin\left(\frac{\pi(2m+1)x'}{a}\right) = \Bigg|$$ using $$\sum_{m=0}^{\infty} \frac{z^{2m+1}}{2m+1} = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

(Jackson, page 75)

$$= \frac{4}{2i} \cdot \frac{1}{2} \frac{1}{\pi} \left[\ln\left(\frac{1+e^{\frac{i\pi x'}{a}}}{1-e^{\frac{i\pi x'}{a}}}\right) - \ln\left(\frac{1+e^{-\frac{i\pi x'}{a}}}{1-e^{-\frac{i\pi x'}{a}}}\right) \right]$$

$$= \frac{-i}{\pi} \ln \frac{\left(1+e^{i\pi\frac{x'}{a}}\right)\left(1-e^{-i\pi\frac{x'}{a}}\right)}{\left(1-e^{i\pi\frac{x'}{a}}\right)\left(1+e^{-i\pi\frac{x'}{a}}\right)} = \Bigg|$$

$$\frac{1+e^{i\pi\frac{x'}{a}}}{1+e^{-i\pi\frac{x'}{a}}} = e^{i\pi\frac{x'}{a}}$$

$$\frac{1-e^{-i\pi\frac{x'}{a}}}{1-e^{i\pi\frac{x'}{a}}} = -e^{-\frac{i\pi x'}{a}}$$

$$= -\frac{i}{\pi}\ln(-1) = -\frac{i}{\pi} i\pi = 1$$ as desired!

$\Rightarrow$ we have shown that

$$\boxed{\frac{2}{a}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right) = \delta(x-x').}$$

$$\Rightarrow \delta^3(\vec{x}-\vec{x}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z') =$$

$$= \frac{8}{abc}\sum_{l,m,n=1}^{\infty} \sin\left(\frac{\pi l x}{a}\right)\sin\left(\frac{\pi l x'}{a}\right)\sin\left(\frac{\pi m y}{b}\right)\sin\left(\frac{\pi m y'}{b}\right)$$

$$\cdot \sin\left(\frac{\pi n z}{c}\right)\sin\left(\frac{\pi n z'}{c}\right)$$