Last time | Fourier integral

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ik\cdot x}}$$ Fourier integral

$$\boxed{A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ik\cdot x}}$$

Orthogonality and completeness of $\{e^{ik\cdot x}\}$ are

$$\boxed{\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')} = \delta(x-x').}$$

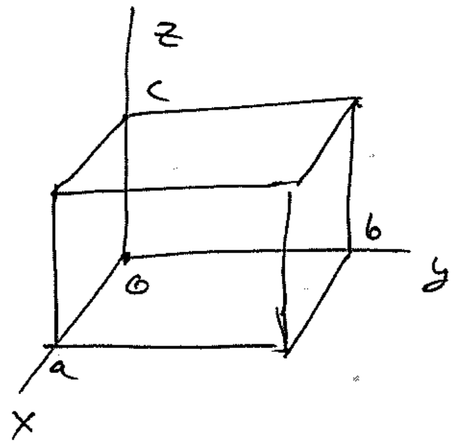
Green function of Poisson eq'n in infinite space:

$$\boxed{G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, \frac{1}{\vec{k}^2}}$$

Laplace & Poisson Equations in rectangular coord's.

(cont'd)

If the problem has rectangular geometry, use separation of variables in rectangular coordinates:



$$\Phi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\begin{cases} X(x) = c_1 e^{i\alpha x} + c_2 e^{-i\alpha x} \\ Y(y) = \bar{c}_1 e^{i\beta y} + \bar{c}_2 e^{-i\beta y} \\ Z(z) = \bar{\bar{c}}_1 e^{\gamma z} + \bar{\bar{c}}_2 e^{-\gamma z} \end{cases}$$

The z-direction is different from x, y $\Rightarrow$

$\Rightarrow$ pick it by convenience. (does not have to be z!)

$$\Rightarrow \gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2} = \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2} = \gamma_{nm}$$

$$\Rightarrow \Phi_{nm}(x,y,z) \propto \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

$$\Rightarrow \Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z).$$

Finally, $\Phi(x,y,z=c) = V(x,y) \Rightarrow$

$$\Rightarrow V(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} c)$$

It's a double Fourier Series $\Rightarrow$ can invert obtaining

$$A_{nm} = \frac{4}{ab\sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy\, V(x,y)\sin(\alpha_n x)\sin(\beta_m y).$$

problem solved!

To find a general solution for Laplace/Poisson equations with Dirichlet boundary conditions ~~~~ we need to find Green function $G_D(\vec{x},\vec{x}')$. The construction is similar to using the Fourier transform: we need to solve $\nabla^2 G_D(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$ and find G_D that vanishes on the boundary.

Method I : expansion in sines.

Need to solve $\nabla'^2 G_D(\vec{x},\vec{x}') = -4\pi\delta^3(\vec{x}-\vec{x}')$ with $G_D(\vec{x},\vec{x}')=0$ for $\vec{x}'$ on the boundary of the box.

Look for G_D in the following form:

$$G_D(\vec{x},\vec{x}') = \sum_{l,m,n=1}^{\infty} G_{lmn}(\vec{x}) \sin\left(\frac{\pi l x'}{a}\right) \sin\left(\frac{\pi m y'}{b}\right) \sin\left(\frac{\pi n z'}{c}\right)$$

$\Rightarrow$ as $G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$ ~ symmetric $\Rightarrow$

$$G_D(\vec{x},\vec{x}') = \sum_{l,m,n=1}^{\infty} \underbrace{\frac{8}{abc} G_{lmn}}_{\text{new coeff.}} \sin\left(\frac{\pi l x}{a}\right) \sin\left(\frac{\pi l x'}{a}\right) \sin\left(\frac{\pi m y}{b}\right) \cdot$$

$$\cdot \sin\left(\frac{\pi m y'}{b}\right) \sin\left(\frac{\pi n z}{c}\right) \sin\left(\frac{\pi n z'}{c}\right).$$

To solve $\nabla'^2 G_D(\vec{x},\vec{x}') = -4\pi\delta^3(\vec{x}-\vec{x}')$ need to expand the δ-functions in sines too.

Above we showed that

$$\{u_n(x)\} = \left\{\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi n x}{a}\right), \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi m x}{a}\right)\right\}$$

is a complete set on $x \in \left(-\frac{a}{2}, \frac{a}{2}\right)$. $\Rightarrow$ it is also a complete set on $x \in (0, a)$. However, on $x \in (0,a)$ we can use a different complete set of functions:

$$\left\{\sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right)\right\}, \quad n = 1, 2, 3, \ldots$$

Clearly the functions are orthogonal:

$$\int_0^a dx \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right) \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n' x}{a}\right) = \delta_{nn'}.$$

To prove completeness write

$$\frac{2}{a}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right) = -\frac{1}{a}\sum_{n=1}^{\infty}\Big[e^{i\frac{\pi n}{a}(x+x')} +$$

$$+ e^{-i\frac{\pi n}{a}(x+x')} - e^{i\frac{\pi n}{a}(x-x')} - e^{-i\frac{\pi n}{a}(x-x')}\Big] =$$

$$= -\frac{1}{a}\sum_{n=-\infty}^{\infty}\left[e^{i\frac{\pi n}{a}(x+x')} - e^{i\frac{\pi n}{a}(x-x')}\right] = \Big|$$ can prove similar to above

$$= \begin{cases} 0, & x+x' \neq 0,\ x-x' \neq 0 \\ \mp\infty, & x+x' = 0 \text{ or } x-x' = 0 \end{cases}$$

& note that $x>0, x'>0$ $\Rightarrow x+x'>0$, never $=0$.

$$\Rightarrow \frac{2}{a}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right) \propto \delta(x-x') \text{ for}$$

$$0 < x, x' < a.$$

To fix the coefficient we integrate:

$$\frac{2}{a}\sum_{n=1}^{\infty}\int_0^a dx \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi n x'}{a}\right) = \sum_{n=1}^{\infty} \frac{-2}{a}\frac{a}{\pi n}\left[\cos(\pi n) - 1\right]$$

$$\cdot \sin\left(\frac{\pi n x'}{a}\right) = -2\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi n}\sin\left(\frac{\pi n x'}{a}\right) = \Big|$$ only odd n survive $n = 2m+1$

$$= 4 \sum_{m=0}^{\infty} \frac{1}{\pi(2m+1)} \sin\left(\frac{\pi(2m+1)x'}{a}\right) =$$

using $\sum_{m=0}^{\infty} \frac{z^{2m+1}}{2m+1} = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$ (Jackson, page 75)

$$= \frac{4}{2i} \cdot \frac{1}{2} \frac{1}{\pi} \left[\ln\left(\frac{1+e^{\frac{i\pi x'}{a}}}{1-e^{\frac{i\pi x'}{a}}}\right) - \ln\left(\frac{1+e^{-\frac{i\pi x'}{a}}}{1-e^{-i\pi\frac{x'}{a}}}\right) \right]$$

$$= \frac{-i}{\pi} \ln \frac{\left(1+e^{i\pi\frac{x'}{a}}\right)\left(1-e^{-i\pi\frac{x'}{a}}\right)}{\left(1-e^{i\pi\frac{x'}{a}}\right)\left(1+e^{-i\pi\frac{x'}{a}}\right)} =$$

$$\frac{1+e^{i\pi\frac{x'}{a}}}{1+e^{-i\pi\frac{x'}{a}}} = e^{i\pi\frac{x'}{a}}$$

$$\frac{1-e^{-i\pi\frac{x'}{a}}}{1-e^{i\pi\frac{x'}{a}}} = -e^{-\frac{i\pi x'}{a}}$$

$$= -\frac{i}{\pi} \ln(-1) = -\frac{i}{\pi} i\pi = 1 \quad \text{as desired !}$$

$\Rightarrow$ we have shown that

$$\boxed{\frac{2}{a} \sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right) = \delta(x-x').}$$

$$\Rightarrow \delta^3(\vec{x}-\vec{x}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z') =$$

$$= \frac{8}{abc} \sum_{\ell,m,n=1}^{\infty} \sin\left(\frac{\pi \ell x}{a}\right) \sin\left(\frac{\pi \ell x'}{a}\right) \sin\left(\frac{\pi m y}{b}\right) \sin\left(\frac{\pi m y'}{b}\right)$$

$$\cdot \sin\left(\frac{\pi n z}{c}\right) \sin\left(\frac{\pi n z'}{c}\right)$$

$$\nabla^2 G_D(\vec{x},\vec{x}') = \sum_{l,m,n=1}^{\infty} \frac{(-8)}{abc} G_{lmn} \left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\pi^2 \cdot$$

$$\cdot \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right) =$$

$$= -4\pi\,\delta^3(\vec{x}-\vec{x}') = -4\pi \frac{8}{abc}\sum_{l,m,n=1}^{\infty}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\cdots$$

$$\Rightarrow \quad G_{lmn} = 4\pi \frac{1}{\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)} = \frac{4}{\pi\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)}$$

similar to $\sim \frac{1}{\vec{k}^2}$ in Fourier space

$$\Rightarrow \quad G_D(\vec{x},\vec{x}') = \frac{32}{\pi abc}\sum_{l,m,n=1}^{\infty}\frac{1}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}} \cdot \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)$$

$$\cdot \sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)$$

Dirichlet Green Function in a box!

$\Rightarrow$ can use it to find potential $\Phi(\vec{x})$ given b.c. on the box.

Method II: separation of variables & expansion in hyperbolic sines.

By analogy with the solution of the problem of a particle in a box, ~~ass~~ look for the Green function in the form:

$$G_D(\vec{x},\vec{x}') = \left(\frac{2}{\sqrt{ab}}\right)^2 \sum_{l,m=1}^{\infty} g_{lm}(z,z') \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \cdot$$

$$\cdot \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right)$$

$$\nabla^2 G_D(\vec{x},\vec{x}') = \frac{4}{ab} \sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \cdot$$

$$\cdot \sin\left(\frac{m\pi y'}{b}\right) \cdot \left\{ \left(-\frac{l^2\pi^2}{a^2} - \frac{m^2\pi^2}{b^2}\right) g_{lm}(z,z') + \frac{\partial^2}{\partial z^2} g_{lm}(z,z') \right\}$$

$$= -4\pi\, \delta^3(\vec{x}-\vec{x}') = -4\pi \frac{4}{ab} \sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right)$$

$$\cdot \sin\left(\frac{m\pi y}{b}\right) \cdot \sin\left(\frac{m\pi y'}{b}\right) \delta(z-z')$$

$$\Rightarrow \frac{\partial^2}{\partial z^2} g_{lm}(z,z') - \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right) g_{lm}(z,z') = -4\pi\, \delta(z-z')$$

$$\Rightarrow \text{define } \varkappa_{lm} = \sqrt{\pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)} \Rightarrow$$

$$\Rightarrow g_{lm}(z,z') = c_1 e^{\varkappa_{lm} z} + c_2 e^{-\varkappa_{lm} z} \quad \text{for, say, } z < z'$$

$\Rightarrow$ as $z < z'$ & $g_{\ell m}(0, z') = 0$ (boundary cond'n)

$\Rightarrow g_{\ell m} \propto \sinh(\varkappa_{\ell m} z)$ for $z < z'$

for $z > z'$: $g_{\ell m}(c, z') = 0 \Rightarrow g_{\ell m} \propto \sinh(\varkappa_{\ell m}(z-c))$

$\Rightarrow$ as $g_{\ell m}(z, z') = g_{\ell m}(z', z) \Rightarrow$

$$g_{\ell m}(z, z') \propto \sinh(\varkappa_{\ell m} z_<) \sinh[\varkappa_{\ell m}(c - z_>)]$$

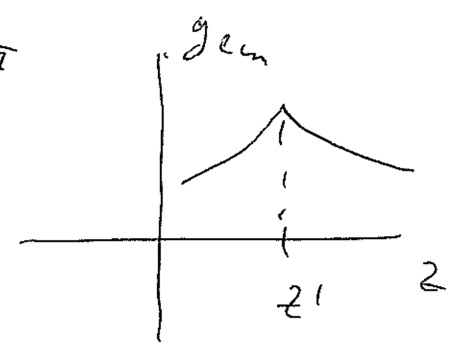
where $z_{\gtrless} = \begin{matrix} \max \\ \min \end{matrix} \{z, z'\}$.

to take into account the δ-fn need to integrate over z in the interval $(z'-\varepsilon, z'+\varepsilon)$

$\Rightarrow g'_{\ell m}(z = z'_+) - g'_{\ell m}(z = z'_-) = -4\pi$

discontinuity in derivative

(a la Schroedinger eqn.)



$$g_{\ell m} = C \sinh(\varkappa_{\ell m} z_<) \sinh[\varkappa_{\ell m}(c - z_>)]$$

$\Rightarrow g'_{\ell m}(z = z'_+) - g'_{\ell m}(z = z'_-) = C(-\varkappa_{\ell m}) \sinh(\varkappa_{\ell m} z')\cdot$

$\cdot \cosh[\varkappa_{\ell m}(c - z')] - C \varkappa_{\ell m} \cosh(\varkappa_{\ell m} z') \sinh[\varkappa_{\ell m}(c - z_>)] =$

$$= -C \varkappa_{lm} \sinh\left[\cancel{\varkappa_{lm} z'} + \varkappa_{lm}(c - \cancel{z'})\right] =$$

$$= -C \varkappa_{lm} \sinh(\varkappa_{lm} c) = -4\pi$$

$$\Rightarrow \quad C = \frac{4\pi}{\varkappa_{lm} \sinh(\varkappa_{lm} c)}$$

$$\Rightarrow \quad G_D(\vec{x}, \vec{x}') = \frac{16\pi}{ab} \sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \cdot$$

$$\cdot \sin\left(\frac{m\pi y'}{b}\right) \sinh(\varkappa_{lm} z_<) \sinh\left[\varkappa_{lm}(c - z_>)\right] \cdot$$

$$\cdot \frac{1}{\varkappa_{lm} \sinh(\varkappa_{lm} c)}$$

An alternative decomposition of Green function.