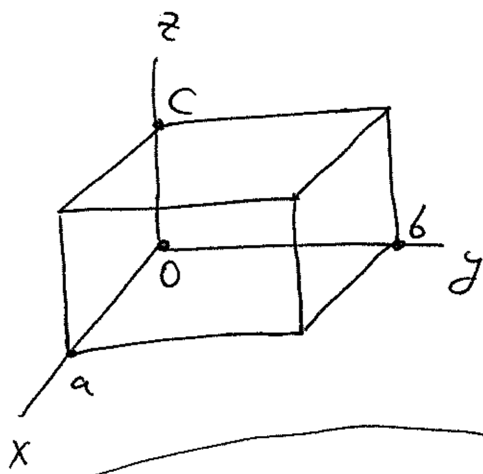
Last time | Constructed Dirichlet Green function for a box as a series in sines:



$$G_0(\vec{x},\vec{x}') = \frac{4}{\pi abc} \sum_{l,m,n=-\infty}^{\infty} \frac{1}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}} \sin\left(\frac{\pi l x}{a}\right) \sin\left(\frac{\pi l x'}{a}\right)$$

$$\cdot \sin\left(\frac{\pi n y}{b}\right) \sin\left(\frac{\pi n y'}{b}\right) \sin\left(\frac{\pi n z}{c}\right) \sin\left(\frac{\pi n z'}{c}\right).$$

We employed the following new representation of δ-function (for functions which are zero for $x=0$ & $x=a$)

$$\delta(x-x') = \frac{1}{a} \sum_{n=-\infty}^{\infty} \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right)$$

Attachement

We know that on interval $x \in [0, a]$

$$\delta(x) = \frac{1}{a} \sum_{n=-\infty}^{\infty} e^{i \frac{2\pi n}{a} x}$$

exponents are complete on that interval.

$\Rightarrow$ changing $a \to 2a$ we'd like to write

$$\text{"} \; \delta(x) = \frac{1}{2a} \sum_{n=-\infty}^{\infty} e^{i \frac{\pi n}{a} x} \; \text{"}.$$

However, since we're still working in $x \in [0, a]$ interval, we have to fix the norm so that

$$\int_0^a dx \; \delta(x) = 1 \quad \Rightarrow \quad \text{in fact}$$

$$\delta(x) = \frac{1}{a} \sum_{n=-\infty}^{\infty} e^{i \frac{\pi n}{a} x}.$$

and, for functions $f(x)$ vanishing at $x = 0$ and $x = a$, $f(0) = f(a) = 0$, we write

$$\delta(x-x') = \frac{1}{a} \sum_{n=-\infty}^{\infty} \sin\left(\frac{\pi n}{a} x\right) \sin\left(\frac{\pi n}{a} x'\right)$$

Method II: separation of variables & expansion in hyperbolic sines.

By analogy with the solution of the problem of a particle in a box, ~~we~~ look for the Green function in the form:

$$G_D(\vec{x}, \vec{x}') = \left(\frac{2}{\sqrt{ab}}\right)^2 \sum_{l,m=1}^{\infty} g_{lm}(z, z') \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \cdot \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right)$$

$$\nabla^2 G_D(\vec{x}, \vec{x}') = \frac{4}{ab} \sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \cdot \sin\left(\frac{m\pi y'}{b}\right) \cdot \left\{\left(-\frac{l^2\pi^2}{a^2} - \frac{m^2\pi^2}{b^2}\right) g_{lm}(z, z') + \frac{\partial^2}{\partial z^2} g_{lm}(z, z')\right\}$$

$$= -4\pi\,\delta^3(\vec{x} - \vec{x}') = -4\pi \frac{4}{ab} \sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \cdot \sin\left(\frac{m\pi y}{b}\right) \cdot \sin\left(\frac{m\pi y'}{b}\right) \delta(z - z')$$

$$\Rightarrow \frac{\partial^2}{\partial z^2} g_{lm}(z, z') - \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right) g_{lm}(z, z') = -4\pi\,\delta(z - z')$$

$\Rightarrow$ define $\varkappa_{lm} = \sqrt{\pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)}$ $\Rightarrow$

$\Rightarrow g_{lm}(z, z') = c_1 e^{\varkappa_{lm} z} + c_2 e^{-\varkappa_{lm} z}$ for, say, $z < z'$

$\Rightarrow$ as $z < z'$ & $g_{\ell m}(0, z') = 0$ (boundary cond'n) (123)

$\Rightarrow g_{\ell m} \propto \sinh(\varkappa_{\ell m} z)$ for $z < z'$

for $z > z'$: $g_{\ell m}(c, z') = 0 \Rightarrow g_{\ell m} \propto \sinh(\varkappa_{\ell m}(z-c))$

$\Rightarrow$ as $g_{\ell m}(z, z') = g_{\ell m}(z', z) \Rightarrow$

$$g_{\ell m}(z, z') \propto \sinh(\varkappa_{\ell m} z_<) \sinh[\varkappa_{\ell m}(c - z_>)]$$

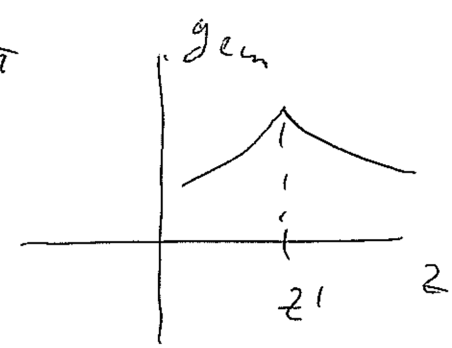
where $z_{\gtrless} = \begin{matrix} \max \\ \min \end{matrix} \{z, z'\}$.

to take into account the δ-ftn need to integrate over z in the interval $(z'-\varepsilon, z'+\varepsilon)$

$\Rightarrow g'_{\ell m}(z = z'_+) - g'_{\ell m}(z = z'_-) = -4\pi$

discontinuity in derivative

(a la Schroedinger eqn.)



$$g_{\ell m} = C \sinh(\varkappa_{\ell m} z_<) \sinh[\varkappa_{\ell m}(c - z_>)]$$

$\Rightarrow g'_{\ell m}(z = z'_+) - g'_{\ell m}(z = z'_-) = C(-\varkappa_{\ell m}) \sinh(\varkappa_{\ell m} z') \cdot$

$\cdot \cosh[\varkappa_{\ell m}(c - z')] - C \varkappa_{\ell m} \cosh(\varkappa_{\ell m} z') \sinh[\varkappa_{\ell m}(c - z_>)] =$

$$= -C\, \varkappa_{lm} \sinh\left[\cancel{\varkappa_{lm} z'} + \varkappa_{lm}(c - z')\right] =$$

$$= -C\, \varkappa_{lm} \sinh\left(\varkappa_{lm} c\right) = -4\pi$$

$$\Rightarrow \quad C = \frac{4\pi}{\varkappa_{lm} \sinh(\varkappa_{lm} c)}$$

$$\Rightarrow \quad G_D(\vec{x}, \vec{x}') = \frac{16\pi}{ab} \sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \cdot$$

$$\cdot \sin\left(\frac{m\pi y'}{b}\right) \sinh\left(\varkappa_{lm} z_<\right) \sinh\left[\varkappa_{lm}(c - z_>)\right] \cdot$$

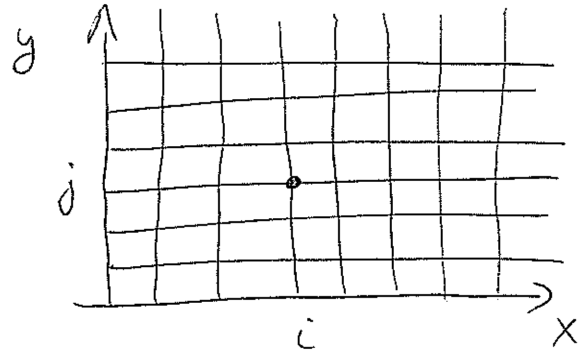
$$\cdot \frac{1}{\varkappa_{lm} \sinh(\varkappa_{lm} c)}$$

an alternative decomposition of Green function.

Numerical Solution of Laplace Equation

~ Relaxation method

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$



$$\frac{\partial \Phi}{\partial x} \to \frac{\Phi(i+1,j) - \Phi(i,j)}{\Delta x}$$ ← lattice spacing

$$\Phi \to \Phi(i,j)$$

$$\left.\begin{aligned} \frac{\partial^2 \Phi}{\partial x^2} &= \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{\Delta x^2} \\ \frac{\partial^2 \Phi}{\partial y^2} &= \frac{\Phi(i,j+1) - 2\Phi(i,j) + \Phi(i,j-1)}{\Delta y^2} \end{aligned}\right\} \text{put } \Delta x = \Delta y = a$$

$$\Rightarrow \nabla^2 \Phi = \frac{1}{a^2}\Big[\Phi(i+1,j) + \Phi(i,j+1) + \Phi(i-1,j) + $$
$$+ \Phi(i,j-1) - 4\Phi(i,j)\Big] = 0$$

average over nearest neighbours

$$\Rightarrow \Phi(i,j) = \frac{1}{4}\Big[\Phi(i+1,j) + \Phi(i,j+1) + \Phi(i-1,j) + \Phi(i,j-1)\Big]$$

Algorithm: (1) Assign random values to Φ on a grid, with Φ on the boundary given by Dirichlet boundary conditions

(2) Average over nearest neighbours until you converge to the answer.

(Improvements: include diagonal neighbours, overrelaxation, etc.)

Same in 3 dim:

$$\nabla^2 \Phi \to \frac{1}{a^2}\Big[\Phi(i+1,j,k) + \Phi(i,j+1,k) + \Phi(i,j,k+1)$$

$$+ \Phi(i-1,j,k) + \Phi(i,j-1,k) + \Phi(i,j,k-1) - 6\,\Phi(i,j,k)\Big]$$

$$= 0$$

$$\Rightarrow \boxed{\Phi(i,j,k) = \frac{1}{6}\Big[\Phi(i+1,j,k) + \Phi(i,j+1,k) + \Phi(i,j,k+1) + \Phi(i-1,j,k) + \Phi(i,j-1,k) + \Phi(i,j,k-1)\Big]}$$

$\Rightarrow$ again, assign random values to $\Phi(i,j,k)$ away from the boundary, and use relaxation method to get $\Phi(i,j,k)$ everywhere $\Rightarrow$ solve Laplace eq'n